2. Let us call a positive integer x a *standalone* prime if x is a prime number, but neither $x-2$ nor $x+2$ is a prime. How many standalone primes x are there between 10 and 35?
3. How many positive integers x are there that so that $x \leq 50$ and there is no nonnegative integer k so that $x = p^k$ for some prime number p?

1.2 When we multiply

In this section, we will be considering the number of all possible choices we can make in a situation when we choose objects with more than one relevant parameter.

1.2.1 The product principle

A car dealership sells five different models, and each model is available in seven different colors. If we are only interested in the model and color of a car, how many different choices does this dealership offer to us?

Let us denote the five models by the capital letters A, B, C, D, and E, and let us denote the seven colors by the numbers 1, 2, 3, 4, 5, 6, and 7. Then each possible choice can be totally described by a *pair* consisting of a capital letter and a number. The list of all choices is shown below.

- $A1, A2, A3, A4, A5, A6, A7$,
- $B1, B2, B3, B4, B5, B6, B7$,
- $C1, C2, C3, C4, C5, C6, C7$,
- $D1, D2, D3, D4, D5, D6, D7$, and
- $E1, E2, E3, E4, E5, E6, E7$.

Here each row corresponds to a certain model. As there are five rows, and each of them consists of seven possible choices, the total number of choices is $5 \times 7 = 35$.

This is an example of the following general theorem.

Theorem 1.6 (Product principle) *Let X and Y be two finite sets. Then the number of pairs (x, y) satisfying $x \in X$ and $y \in Y$ is $|X| \times |Y|$.*

Proof: There are $|X|$ choices for the first element x of the pair (x, y); then, regardless of what we choose for x, there are $|Y|$ choices for y. Each choice of x can be paired with each choice of y, so the statement is proved. $\diamondsuit$

Note that the set of all ordered pairs (x, y) so that $x \in X$ and $y \in Y$ is called the *direct product* (or Cartesian product) of X and Y, and is often denoted by $X \times Y$. We call the pairs (x, y) *ordered pairs* because the order of the two elements matters in them. That is, $(x, y) \neq (y, x)$, unless $x = y$.

Example 1.7 *The number of two-digit positive integers is 90.*

Solution: Indeed, a two-digit positive integer is nothing but an ordered pair (x, y), where x is the first digit and y is the second digit. Note that x must come from the set $X = \{1, 2, \cdots, 9\}$, while y must come from the set $Y = \{0, 1, \cdots, 9\}$. Therefore, $|X| = 9$ and $|Y| = 10$, and the statement is proved by Theorem 1.6. $\Diamond$

Theorem 1.8 (Generalized product principle) *Let $X_1, X_2, \cdots, X_k$ be finite sets. Then, the number of k-tuples $(x_1, x_2, \cdots, x_k)$ satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times \cdots \times |X_k|$.*

Informally, we could argue as follows. There are $|X_1|$ choices for x_1; then, regardless of the choice made, there are $|X_2|$ choices for x_2, so by Theorem 1.6, there are $|X_1| \times |X_2|$ choices for the sequence (x_1, x_2). Then there are $|X_3|$ choices for x_3, so again by Theorem 1.6, there are $|X_1| \times |X_2| \times |X_3|$ choices for the sequence (x_1, x_2, x_3). Continuing this argument until we get to x_k proves the theorem.

The line of thinking in this argument is correct, but the last sentence is somewhat less than rigorous. In order to obtain a completely formal proof, we will use the method of mathematical induction. It is very likely that the reader has already seen that method. A brief overview of the method can be found in the appendix.

Proof: (of Theorem 1.8) We prove the statement by induction on k. For $k = 1$, there is nothing to prove, and for $k = 2$, the statement reduces to the product principle.

Now let us assume that we know the statement for $k-1$, and let us prove it for k. A k-tuple $(x_1, x_2, \cdots, x_k)$ satisfying $x_i \in X_i$ can be decomposed into an ordered pair $((x_1, x_2, \cdots, x_{k-1}), x_k)$, where we still have $x_i \in X_i$. The number of such $(k-1)$-tuples $(x_1, x_2, \cdots, x_{k-1})$ is, by our induction hypothesis, $|X_1| \times |X_2| \times \cdots \times |X_{k-1}|$. The number of elements $x_k \in X_k$ is $|X_k|$. Therefore, by the product principle, the number of ordered pairs $((x_1, x_2, \cdots, x_{k-1}), x_k)$ satisfying the conditions is

$$(|X_1| \times |X_2| \times \cdots \times |X_{k-1}|) \times |X_k|,$$

so this is also the number of k-tuples $(x_1, x_2, \cdots, x_k)$ satisfying $x_i \in X_i$. $\Diamond$