Quick Check

1. What is the number of subsets of [10] that have at least four elements, but not more than six elements?
2. A company has 30 employees, 20 of whom work in production, and ten of whom work in sales. How many ways are there to form a committee of six employees of this company so that four of the committee members work in production and two committee members work in sales?
3. Continuing the previous question, how many ways are there to form a six-member committee if the committee is to have *at least* one member who works in production and *at least* one member who works in sales?

1.4 Applications of basic counting principles

We will now consider more complex situations where the basic counting principles turn out to be useful.

1.4.1 Bijective proofs

The following example will teach us an extremely important proof technique in enumerative combinatorics.

Example 1.26 *In a certain part of our town, the streets form a square grid, and each street is one way to the north or to the east. Let us assume that our car is currently at the southwest corner of this grid, which we will denote by* $O = (0, 0)$.

(a) In how many ways can we drive to the point $X = (6, 4)$*?*

(b) In how many ways can we drive to the point X *if we want to stop at the bakery at* $Y = (4, 2)$*?*

(c) In how many ways can we drive to the point X *if we want to stop at either the ice cream shop at* $U = (3, 2)$ *or at the coffee shop at* $V = (2, 3)$*?*

See Figure 1.3 for an illustration.

Solution:

(a) The car needs to travel ten blocks, namely, six blocks to the east and four blocks to the north. In other words, the driver needs to choose a six-element subset of [10] that will tell him when to drive east.

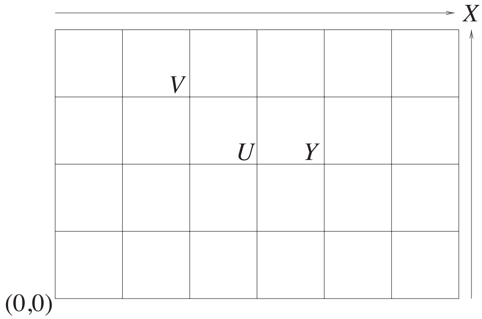

Figure 1.3
The grid of our one-way streets.

If, for instance, he chooses the set $\{2, 3, 5, 7, 8, 9\}$, then his second, third, fifth, seventh, eighth, and ninth streets will go east, and all the rest, that is, his first, fourth, sixth, and tenth streets will go north. Since the number of six-element subsets of [10] is $\binom{10}{6}$, this is the number of ways the car can get to point X.

(b) The car first needs to get to Y. By the argument of part (a), there are $\binom{6}{4} = 15$ ways to do this. Then, it needs to go from Y to X, and, by an analogous argument, there are $\binom{4}{2} = 6$ ways to do it. As any path from $(0, 0)$ to Y can be followed by any path from Y to X, the total number of acceptable paths is $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$.

(c) Using the argument explained in part (b), there are $\binom{5}{3}\binom{5}{3} = 100$ paths from $(0, 0)$ to X via U, and there are $\binom{5}{2}\binom{5}{4} = 50$ paths from $(0, 0)$ to X via V. Note that no path can go through both U and V. Therefore, the total number of acceptable paths is $\binom{5}{3}\binom{5}{3} + \binom{5}{2}\binom{5}{4} = 150$.

◊

Let us analyze the argument of the above example in detail. In part (a), we had to count the possible paths from $(0, 0)$ to $(6, 4)$. We said that this was the same as counting the six-element subsets of [10]. Why could we say that? We could say that because there is a *one-to-one* correspondence between the set S of our lattice paths and the set T of six-element subsets of [10]. Therefore, we must have $|S| = |T|$. Note that this is the special case of Theorem 1.21 (the division principle) when $d = 1$.

The special case of $d = 1$ of the division principle is so important that it has its own name.

Definition 1.27 *If the map $f : S \to T$ is one-to-one and onto, then we call f a* bijection.

In other words, if $f : S \to T$ is a bijection, then it creates pairs, matching each element of S to a different element of T.

Corollary 1.28 *Let S and T be finite sets. If a bijection $f : S \to T$ exists, then $|S| = |T|$.*

Note that the requirement that S and T be finite can be dropped, and then one can *define* the notion that two infinite sets have the same size if there is a bijection between them. This is a very interesting topic, but it belongs to a textbook on Set Theory; therefore, we will not discuss it here.

See Figure 1.4 for the diagram of a generic bijection.

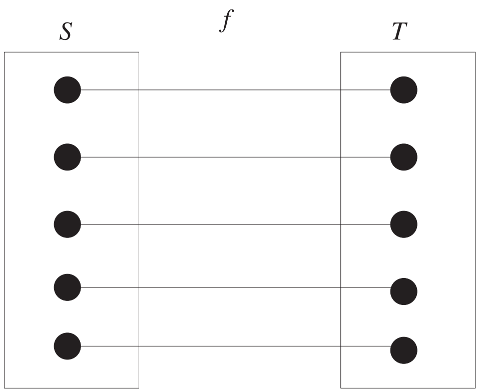

Figure 1.4
The diagram of a generic bijection.

The idea of bijections, or bijective proofs, is used very often in counting arguments. When we want to enumerate elements of a set S, we can instead prove that a bijection $f : S \to T$ exists with some set T whose number of elements we know. Usually, this is done by first defining a function f, then showing that this f is indeed a bijection from S into T. Once that is done, we can conclude that $|S| = |T|$. Sometimes, we do not need the actual number of elements in the sets, just the fact that the two sets have the same number of elements. In that case, the method of bijections can save us the actual counting. The reader will see many examples of this method in the next section, and the rest of the book for that matter. For now, let us see some simple applications of the method.

Proposition 1.29 *For any positive integer n, the number of divisors of n that are larger than $\sqrt{n}$ is equal to the number of divisors of n that are smaller than $\sqrt{n}$.*

As this is our first bijective proof, we will explain it in full detail. In particular, we will show how one can prove that a map is indeed a bijection. The reader should not be discouraged by thinking that this proof is too technical. All we do is show that two sets have the same size by matching their elements, one by one.

Proof: Let S be the set of divisors of n that are larger than $\sqrt{n}$, and let T be the set of divisors of n that are smaller than $\sqrt{n}$. Define $f : S \rightarrow T$ by $f(s) = n/s$.

Now comes that heart of the proof, that is, we will show that f is *indeed a bijection from S into T*. First, for all $s \in S$, the equality

$$s \cdot f(s) = n \tag{1.6}$$

holds, so $f(s)$ is indeed a divisor of n. Second, $f(s) < \sqrt{n}$ must hold, otherwise $s \cdot f(s) > \sqrt{n} \cdot \sqrt{n} = n$, contradicting (1.6). Therefore, $f(s)$ is always an element of T, and so f is indeed a function from S into T.

Now we have to show that f is one-to-one, that is, for all $t \in T$, there exists exactly one $s \in S$ so that $f(s) = t$. On one hand, there is at least one such s, namely, $s = n/t$. Indeed, by the definition of f, we have $f(n/t) = \frac{n}{n/t} = t$. On the other hand, this is the only good s. Indeed, if $f(s) = t$, then by (1.6), we must have $s \cdot t = n$, so $s = n/t$. Therefore, f is a bijection, and so S and T have the same number of elements. $\Diamond$

Example 1.30 *The integer 1000 has exactly eight divisors that are larger than $\sqrt{1000}$.*

Solution: By Proposition 1.29, it suffices to count the divisors of 1000 that are *smaller* than $\sqrt{1000} = 31.62$. They are 1, 2, 4, 5, 8, 10, 20, and 25, so there are indeed eight of them. $\Diamond$

In other words, instead of scanning the interval $[32, 1000]$ for divisors, we only had to scan the much shorter interval $[1, 31]$.

The way in which we showed that the function f was indeed a bijection from S into T in the above example is fairly typical. Let us summarize this method for future reference.

In order to prove that $|S| = |T|$ by the method of bijections, proceed as follows:

1. Define a function f on the set S that has a chance to be a bijection from S into T.
2. Show that for all $s \in S$, the relation $f(s) \in T$ holds.
3. Show that for all $t \in T$, there is exactly one $s \in S$ satisfying $f(s) = t$. This is often done in two smaller steps, namely,

(a) proving that there is at least one s satisfying $f(s) = t$, and
(b) proving there is at most one s satisfying $f(s) = t$.

Example 1.31 *A new house has ten rooms. For each room, the owner can decide whether he wants a ceiling fan for that room, and if yes, whether he wants it to be a cheaper or a more expensive model. How many different possibilities does the owner have for all ten rooms?*

Solution: We claim that the number of possibilities is 3^{10}. We show this by constructing a bijection f from the set S of all possibilities the owner has to the set T of all ten-letter words over the alphabet $\{a, b, c\}$. Then our result will follow from Corollary 1.11.

Let $s \in S$, and define the ith letter of $f(s)$ by

$$f(s)_i = \begin{cases} a \text{ if there is no ceiling fan in room } i, \\ b \text{ if there is a cheap ceiling fan in room } i, \\ c \text{ if there is a more expensive ceiling fan in room } i. \end{cases}$$

Our construction then implies that $f(s) \in T$, so f is indeed a function from S to T. Furthermore, f is indeed a bijection, as any word t in T tells us with no ambiguity what kind of ceiling fan each room needs to have in s if $f(s) = t$. $\Diamond$

1.4.1.1 Catalan numbers

For a more involved bijection, let us return to Example 1.26. A drive that is allowed in that example is called a *northeastern lattice path*. Northeastern lattice paths are omnipresent in enumerative combinatorics, since they can represent a plethora of different objects.

Lemma 1.32 *The number of northeastern lattice paths from* $(0, 0)$ *to* (n, n) *that never go above the diagonal* $x = y$ *(the main diagonal) is equal to the number of ways to fill a* $2 \times n$ *grid with the elements of* $[2n]$ *using each element once so that each row and column is increasing (to the right and down).*

For brevity, a $2 \times n$ rectangle whose boxes contain the elements of $[2n]$ so that each element is used once and each row and column is increasing (to the right and down) will be called a *Standard Young Tableau* of shape $2 \times n$.

Example 1.33 *Let* $n = 3$*. Then, there are five northeastern lattice paths from* $(0, 0)$ *to* $(3, 3)$ *that do not go above the main diagonal; they are shown in Figure 1.5. There are five Standard Young Tableaux of shape* 2×3*, which are shown in Figure 1.6.*

Proof: (of Lemma 1.32) It should not come as a surprise that we will construct a bijection from the set S of all northeastern lattice paths from $(0, 0)$ to (n, n)

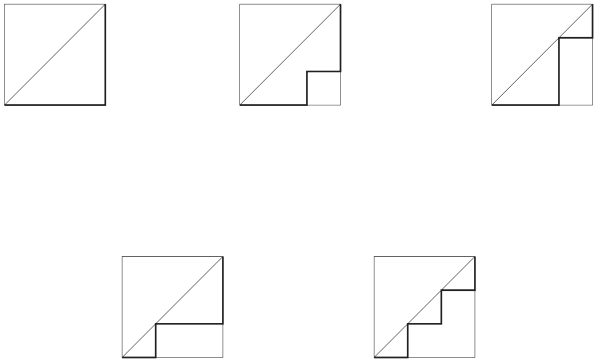

Figure 1.5
The five northeastern lattice paths that do not go above the main diagonal, for $n = 3$.

that do not go above the main diagonal into the set T of all Standard Young Tableaux of shape $2 \times n$.

Our bijection f is defined as follows: Let $s \in S$. Let $e_1, e_2, \cdots, e_n$ denote the positions of the n east steps of f. That is, if the first three east steps of f are in fact the first, third, and fourth steps of f (and so the second, fifth, and sixth steps of f are to the north), then $e_1 = 1$, $e_2 = 3$, and $e_3 = 4$. Similarly, let $n_1, n_2, \cdots, n_n$ denote the north steps of s. That is, keeping our previous example, $n_1 = 2$, $n_2 = 5$, and $n_3 = 6$.

Let $f(s)$ be the array of shape $2 \times n$ whose first row is $e_1, e_2, \cdots, e_n$ and whose second row is $n_1, n_2, \cdots, n_n$. We claim that $f(s) \in T$. The rows of $f(s)$ are increasing, since the ith east step had to happen before the $(i+1)$st east step, and the same holds for north steps. We claim that the columns are increasing as well. Indeed, otherwise $n_j < e_j$ would hold for some j, meaning

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

Figure 1.6
The five Standard Young Tableaux of shape $2 \times n$.