

**Quick Check**

1. A frog is sitting on the number line at  $x = 0$ . At any point of time, the frog can jump one unit to the right, or one unit to the left, but it can never go to the left of  $x = 0$ . Prove that the number of sequences of  $2n$  jumps that end at  $x = 0$  is equal to the number of  $2 \times n$  Standard Young Tableaux.
2. Prove the identity

$$\sum_{k=0}^n \binom{n}{k} 2^k \cdot (-1)^{n-k} = 1.$$

3. What is the number ways to list the entries 1, 1, 1, 2, 2, and 3 so that the first entry of the list is not the same digit as the last entry of the list?

**1.5 The pigeonhole principle**

The pigeonhole principle is almost as simple to state as the addition principle. However, it turns out to be a very powerful tool that has a plethora of surprisingly strong applications.

**Theorem 1.44 (Pigeonhole principle)** *Let  $A_1, A_2, \dots, A_k$  be finite sets that are pairwise disjoint. Let us assume that*

$$|A_1 \cup A_2 \cup \dots \cup A_k| > kr.$$

*Then there exists at least one index  $i$  so that  $|A_i| > r$ .*

In other words, if the union of a few disjoint sets is “large,” then at least one of those sets must also be “quite large.” This is something you have probably experienced when scheduling classes for a busy week. If you want to schedule more than 20 hours of classes for a five-day week, then you will have more than four hours of classes on at least one day.

A classic way of thinking about the pigeonhole principle is by boxes and balls (which seems more humane than putting pigeons into holes). If we distribute more than  $kr$  balls in  $k$  boxes, then at least one box will have more than  $r$  balls.

The proof of the pigeonhole principle is an example of a standard proof technique. We prove the theorem by showing that its opposite is impossible. That is, we assume that the opposite of our statement is true (that is, in this case, we assume that there is no index  $i$  so that  $|A_i| > r$  holds), and

we derive a contradiction from this assumption. This procedure is called an *indirect proof* or *proof by contradiction*.

**Proof:** Let us assume that the statement we want to prove is false. Then  $|A_i| \leq r$  holds for each  $i$ . Therefore,

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_n| \leq kr,$$

which contradicts our original assumption that  $|A_1 \cup A_2 \cup \cdots \cup A_k| > kr$ .  $\diamond$

**Example 1.45** *There are at least eight people currently living in New York City who were born in the same hour of the same day of the same year.*

While the population of New York City keeps changing day by day, it is safe to assume that it is always over 7.5 million people.

**Solution:** We can safely assume that all residents of New York City are no more than 120 years old. Therefore, they were born at most 120 years ago. As each year consists of at most 366 days, the people we are considering are at most  $120 \cdot 366 = 43920$  days old. The number of hours in that many days is  $k = 24 \cdot 43920 = 1054080$ .

Now let  $A_1$  be the set of NYC residents who were born at the first eligible hour, let  $A_2$  be the set of NYC residents who were born at the second eligible hour, and so on, with  $A_k$  denoting the set of NYC residents who were born in the last eligible hour (that is, the hour that is ending now). Then we know that

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \geq 7500000, \tag{1.11}$$

because the union of all the  $A_i$  is the population of New York City. Let us now apply the pigeonhole principle with  $r = 7$ . As  $k = 1054080$ , we see that the left-hand side is larger than  $7k$ , so at least one of the  $A_i$  must contain more than seven people.  $\diamond$

A frequently applied special case of the pigeonhole principle is when  $r = 1$ . In that case, the principle says that if  $k$  boxes altogether contain more than  $k$  balls, then at least one box has to contain more than one ball. Even this simple special case has interesting applications, as we will see below.

**Example 1.46** *Consider the sequence  $1, 3, 7, 15, 31, \dots$ , in other words, the sequence whose  $i$ th element is  $a_i = 2^i - 1$ . Let  $q$  be any odd integer. Then our sequence contains an element that is divisible by  $q$ .*

This is a rather strong statement. We did not say *anything* about  $q$  other than it is odd. Therefore, our statement holds true for  $q = 17$  just as much as for  $q = 2007$ , or  $q = 3542679$ . All these numbers have a multiple that is one less than a power of two.

**Solution:** Consider the first  $q$  elements of our sequence. If one of them is divisible by  $q$ , then we are done. If not, then consider their remainders modulo  $q$ . That is, let us write

$$a_i = d_i q + r_i,$$

where  $0 < r_i < q$ , and  $d_i = \lfloor a_i/q \rfloor$ . As the integers  $r_1, r_2, \dots, r_q$  all come from the open interval  $(0, q)$ , there are  $q - 1$  possibilities for their values. On the other hand, their number is  $q$ , so, by the pigeonhole principle, there have to be two of them that are equal. Say these are  $r_n$  and  $r_m$ , with  $n > m$ . Then  $a_n = d_n q + r_n$  and  $a_m = d_m q + r_m$ , so

$$a_n - a_m = (d_n - d_m)q$$

or, after rearranging,

$$\begin{aligned} (d_n - d_m)q &= a_n - a_m \\ &= (2^n - 1) - (2^m - 1) \\ &= 2^m(2^{n-m} - 1) \\ &= 2^m a_{n-m}. \end{aligned}$$

As the first expression of our chain of equations is divisible by  $q$ , so too must be the last expression. Note that  $2^{n-m}$  is relatively prime to any odd number  $q$ , that is, the largest common divisor of  $2^{n-m}$  and  $q$  is 1. Therefore, the equality  $(d_n - d_m)q = 2^{n-m} a_{n-m}$  implies that  $a_{n-m}$  is divisible by  $q$ . This completes the solution.  $\diamond$

In what follows, we will write  $[n]$  for the set  $\{1, 2, \dots, n\}$ , that is, the set of the first  $n$  positive integers.

**Example 1.47** *Let us arbitrarily select  $n + 1$  distinct integers from the set  $[2n]$ . Then there is at least one pair of selected integers whose sum is  $2n + 1$ , and there is at least one pair of selected integers whose difference is  $n$ .*

**Solution:** Let us split our set into  $n$  subsets, namely, the subset  $\{1, 2n\}$ , the subset  $\{2, 2n - 1\}$ , and so on, the generic subset being  $\{i, 2n + 1 - i\}$ , where  $1 \leq i \leq n$ . As we have selected  $n + 1$  integers and have split  $[2n]$  into only  $n$  two-element subsets, the pigeonhole principle implies that there has to be a two-element subset  $X$  so that both elements of  $X$  have been selected. The sum of the elements of  $X$  is  $2n + 1$ ; therefore, our first claim is proved.

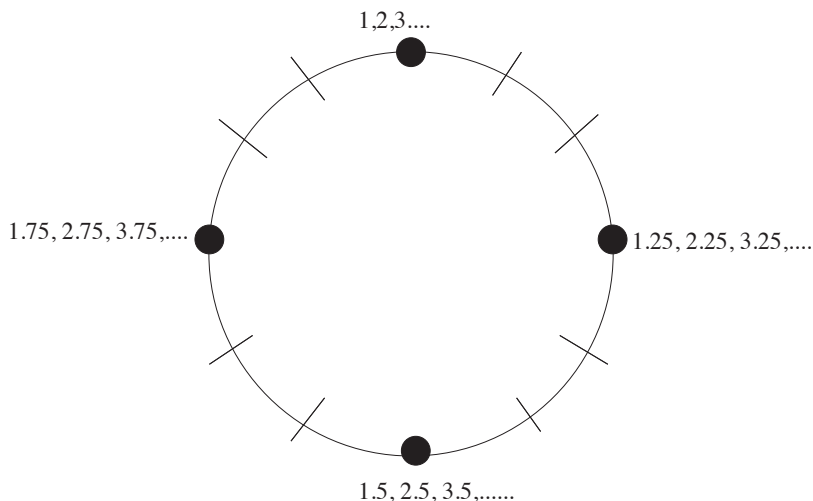
Now let us split  $[2n]$  into the  $n$  subsets  $\{1, n + 1\}$ ,  $\{2, n + 2\}$ , and so on, the generic subset being  $\{i, n + i\}$ , where  $1 \leq i \leq n$ . Again, by the pigeonhole principle, one of these  $n$  subsets, say  $Y$ , must consist of two selected integers. However, the difference between the two elements of  $Y$  is  $n$ , and our second claim is proved.  $\diamond$

Based on the previous two examples, the reader might think that the pigeonhole principle can only be applied to problems in which all relevant objects are *integers*. This is far from true, as shown by the following example.

**Example 1.48** *Let  $p$  be any positive irrational number. Then there exists a positive integer  $n$  so that the distance between  $np$  and the closest integer is less than  $10^{-10}$ .*

There is nothing magical about  $10^{-10}$  here, we just chose it to represent “tiny number.” In other words, the example claims that the multiples of any irrational number can get as close to an integer as we like. This is often expressed by saying that the set of irrational real numbers is *dense* within the set of all real numbers.

**Solution:** Let us represent the set of all positive real numbers by a *circle*, as shown in Figure 1.11. That is, we think of the circle as having circumference 1, and two real numbers are represented by the same point on the circle if their difference is an integer.



**Figure 1.11**

A circle representing  $\mathbf{R}^+$ .

Now let us divide the perimeter of the circle into  $10^{10}$  equal parts. (Figure 1.11 shows a subdivision to 12 parts for better picture quality.) Let us take the first  $10^{10} + 1$  elements of the sequence  $p, 2p, 3p, \dots$ . Since there are only  $10^{10}$  arcs, by the pigeonhole principle there has to be one arc  $T$  that contains at least two elements of this sequence. Let  $ip$  and  $jp$  be two such elements, with  $i < j$ . Then this implies that the distance from  $(j - i)p$  to the nearest integer is less than the length of the arc  $T$ , that is,  $10^{-10}$ .  $\diamond$