(D) Proof techniques

1. Combinatorial proof: Show that two expressions A and B are equal by showing that they count the number of elements of the same set.
2. Bijective proof: Show that the finite sets S and T have the same number of elements by constructing a bijection $f : S \to T$.
3. Indirect proof: Show that a statement is true by assuming that its contrary is true and deducing a contradiction from that assumption.

1.8 Exercises

1. An airline has two ticket counters. At a given point of time, n people stand in line at the first counter and m people stand in line at the second counter.

   (a) Let us assume that the lines are not necessarily formed on a first come, first serve basis. How many ways are there to rearrange the two lines if nobody is allowed to switch counters?

   (b) Use your answer of part (a) to justify the identity $0! = 1$.

2. Justify the convention $m^0 = 1$ in a way similar to the previous exercise.
3. Find the number of ways to place n rooks on an $n \times n$ chess board so that no two of them attack each other.
4. How many ways are there to place *some* rooks on an $n \times n$ chess board so that no two of them attack each other?
5. A long-distance running race had 15 participants, among them Amy and Bob. How many outcomes are possible if we know that Amy finished ahead of Bob?
6. In one of the Florida lottery games, one has to match six numbers out of 49 numbers to win. How many lottery tickets does one have to buy in order to be sure to have a perfect match?
7. How many four-digit positive integers are there that contain the digit 3 and are divisible by 5?
8. The door of an apartment building can only be opened by a four-digit code. A user forgot the code, but remembers that it uses each of the digits 3, 5, and 9, and no other digits. In the worst case, how many attempts will it take for the user to get into the building?

9. How many ways are there to list the digits 1, 1, 2, 2, 3, 4, 5 so that the digits 3 and 4 are not in consecutive positions?
10. A student works in a bookstore where he is required to work at least four and at most five days a week, at least one of which has to be a weekend day (Saturday or Sunday). How many different weekly work schedules can this student have?
11. A college football coach must choose four new players who will receive scholarships. He can choose among 20 incoming players, half of whom are offensive players, and half of whom are defensive players. The coach can award the scholarships in any way, as long as at least one offensive player and at least one defensive player gets a scholarship. How many possibilities does the coach have?
12. Consider the square grid we discussed in Example 1.26. How many ways are there to drive from $O = (0,0)$ to $A = (6,6)$ if we have to stop at $B = (4,4)$, but also must avoid $C = (3,1)$?
13. Prove formula (1.7).
14. Prove that for all positive integers k and n, with $k \leq n$,

$$\binom{n}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{n-1}{k-1}.$$

15. +[1] Let n, p, and q be fixed positive integers, so that $p \leq n$, and $q \leq n$. Prove the identity

$$\binom{n}{p}\binom{n}{q} = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{p-k}\binom{n-k}{q-k}.$$

16. Let $n = 4k+2$, for some nonnegative integer k. Prove that exactly $1/4$ of all subsets of $[n]$ have a size that is divisible by four.
17. Find a closed formula for the expression

$$\sum_{k=0}^{n} \binom{n}{k} 4^k (-1)^{n-k}.$$

18. A basketball fan looked at the newspaper for a short time and checked the standings of the seven-team division of his favorite team. Later, he tried to remember the standings, but he did not recall every detail. However, he recalled that the Lakers were in the first position, and the Sonics in the fifth position. Furthermore, he remembered that the Kings were ahead of the Trailblazers, who in turn were ahead of the Clippers. How many possibilities does that leave open for the complete standings of this division?

[1] The plus sign indicates an advanced problem.

19. The basketball fan of the previous exercise tried to remember the standings of another seven-team division, but some details escaped him again. All he could remember was that the Rockets and the Grizzlies were in consecutive positions (but he forgot in which order), and the Nuggets were behind the Spurs. How many possibilities does that leave open for the complete standings of this division?

20. Let us revisit Example 1.43 so that now we do take the venues of the games into account. That is, of the four games the Magic play against each team of its division, two have to be played at home, and two have to be played away. Of the two games played against each team of the other conference, one has to be at home, one away. Finally, the Magic can choose five of the remaining opponents whom they will play twice at home, once away, and then they will play the remaining four opponents twice away, and once at home. How many different schedules can the Orlando Magic have?

21. How many ways are there to choose subsets S and T of $[n]$ if there are no conditions whatsoever imposed on these subsets?

22. (a) How many ways are there to choose subsets S and T of $[n]$ so that S contains T?

    (b) How many ways are there to choose *disjoint* subsets R and U of $[n]$?

23. We want to form n pairs from $2n$ tennis players, to play n games. In how many ways can we do this?

24. Let $r > e$. Prove that for all $n \geq 1$, the inequality

$$n! > \left(\frac{n}{r}\right)^n$$

holds. Do not use Stirling's formula. You may want to use the fact learned in calculus that the sequence $a_n = (n/(n+1))^n$ is monotone decreasing and converges to $1/e$.

25. + Find an explicit formula for the Catalan numbers, defined (after Example 1.34) as follows:

    (a) Note that, by the subtraction principle, the number c_n is equal to the number a_n of all northeastern lattice paths from $(0, 0)$ to (n, n) minus the number b_n of northeastern lattice paths from $(0, 0)$ to (n, n) that go above the main diagonal at some point. Then find an explicit formula for the numbers a_n.

    (b) Find a bijection between the set S of lattice paths enumerated by b_n and the set T of all northeastern lattice paths from $(-1, 1)$ to (n, n).

    (c) Find a formula for $|T|$ and apply the subtraction principle.

26. + Let k be a positive integer. Let (a, b) be a point in the first quadrant on or below the line $x = ky$. Prove that the number of northeastern lattice paths from $(0, 0)$ to (a, b) that do not *touch* the line $x = ky$, except for their origin, is

$$\binom{a+b}{a}\frac{a-kb}{a+b}.$$

27. Find a closed formula for the expression

$$\sum_{k=1}^{n}\frac{k}{n^k}\binom{n}{k}.$$

28. + Prove the identity

$$n = \sum_{k=1}^{n}\frac{k}{n^k}\binom{n+1}{k+1}.$$

29. A tennis tournament has 85 participants. Players who lose a game are immediately eliminated; players who win a game keep playing. Still, the organizers have a lot of choices to make. They could give a first-round bye to some players so that after the first round there are $2^6 = 64$ players and no more byes are needed. Or they could give even a second-round bye to the best players, or possibly even a third-round bye to the very best ones. What is the best strategy for the organizers if they want to choose the winner of the tournament using as few games as possible?

30. Sixteen players participated in a round-robin tennis tournament. Each of them won a different number of games. How many games did the player who finished sixth win?

31. Let $A_1, A_2, \cdots, A_k$ be finite sets that are not necessarily pairwise disjoint. Prove that

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \leq |A_1| + |A_2| + \cdots + |A_k|.$$

32. Prove that the pigeonhole principle holds, even if we do not assume that the sets $A_1, A_2, \cdots, A_k$ are pairwise disjoint.

33. + All points of the plane that have integer coordinates are colored red, blue, or green. Prove that there will be a rectangle whose vertices are all of the same color.

34. A computer program generated 175 positive integers at random, none of which had a prime divisor larger than 10. Prove that we can always find three numbers among them whose product is the cube of an integer.