1.9 Solutions to exercises

1. There are $n!$ ways for people at the first counter to form a line, and $m!$ ways for people at the second counter to form a line. Therefore, by the product principle, there are altogether $n!m!$ possibilities. Now, if $m = 0$, then the number of total possibilities is $n!$, as people at the first counter still have the same number of ways to form a line. Therefore, $m! = 0! = 1$ is the only choice that leaves the formula $n!m!$ valid.
2. Let us count words consisting of $a + b$ letters, the first a of which comes from an alphabet of n letters, and the last b of which comes from an alphabet of m letters. By the product principle, the number of these words is $n^a m^b$. In particular, if $b = 0$, then we just count a-letter words over an n-element alphabet. We know that the number of these is n^a. Therefore, in this case $n^a = n^a \cdot m^0$, yielding $m^0 = 1$.
3. There has to be one rook in each column. The first rook can be anywhere in its column (n possibilities). The second rook can be anywhere in its column except in the same row where the first rook is, which leaves $n - 1$ possibilities. The third rook can be anywhere in its column, except in the rows taken by the first and second rooks, which leaves $n-2$ possibilities, and so on, leading to $n \cdot (n-1) \cdots 2 \cdot 1 = n!$ possibilities.
4. If we place k rooks, then we first need to choose the k columns in which these rooks will be placed. We can do that in $\binom{n}{k}$ ways. Continuing the line of thought of the solution of the previous exercise, we can then place our k rooks into the chosen columns in $(n)_k$ ways. Therefore, the total number of possibilities is
$$\sum_{k=1}^{n} \binom{n}{k} (n)_k.$$
5. The number of possible outcomes is $15!/2$. Indeed, let S be the set of possible outcomes, and let T be the set of impossible outcomes, that is, those in which Bob finishes ahead of Amy. Then there is a bijection $f : S \to T$, namely, the function that simply switches Amy and Bob.
6. As the order of the numbers played does not matter, this problem simply asks for the number of all six-element subsets of [49]. As we know from Theorem 1.4, this is $\binom{49}{6} = 13983816$.
7. The number of all four-digit integers divisible by five is $9 \cdot 10 \cdot 10 \cdot 2 = 1800$, since such an integer has to end in 0 or 5. Among these, $8 \cdot 9 \cdot 9 \cdot 2 = 1296$ do not contain the digit 3, so $1800 - 1296 = 504$ numbers do contain the digit 3.

8. There will have to be one digit that is used twice, and the remaining two digits are used once. If we used the digit 3 twice, then we have to make $\binom{4}{2,1,1} = 12$ attempts. By symmetry, the same argument applies if the digit 5 or the digit 9 is repeated. Therefore, in the worst case, we will need to make $3 \cdot 12 = 36$ attempts.
9. Without the requirement that 3 and 4 are not in consecutive positions, there would be $\binom{7}{2,2,1,1,1} = 1260$ ways to list the given digits. Let us count those lists in which 3 and 4 are in consecutive positions. To do this, let us *glue* 3 and 4 together to get a superdigit, X. Then there are $\binom{6}{2,2,1,1} = 180$ possible lists. However, each of them corresponds to two original lists because X can be replaced by 34 or 43. Therefore, by the product principle, there are $2 \cdot 180 = 360$ bad lists, and so, by the subtraction principle, there are $1260 - 360 = 900$ good lists.
10. There are $\binom{7}{4} + \binom{7}{5} = 35 + 21 = 56$ ways to choose four or five days of a week. Of these, $\binom{5}{4} + \binom{5}{5} = 5 + 1 = 6$ will not contain any weekend days. Therefore, the number of possible schedules is $56 - 6 = 50$.
11. With no restrictions, the coach would have $\binom{20}{4}$ possibilities. The only bad choices are when all scholarships go to offensive players, or when all scholarships go to defensive players. Therefore, the number of bad choices is $2 \cdot \binom{10}{4}$, and so, by the subtraction principle, the number of good choices is $\binom{20}{4} - 2 \cdot \binom{10}{4}$.
12. Just as in Example 1.26, the number of ways to drive from O to A via B is the number of ways to drive from O to B times the number of ways to drive from B to A. In other words, it is $\binom{8}{4} \cdot \binom{4}{2} = 70 \cdot 6 = 420$. Now we have to subtract the number of bad paths among the paths we just counted, that is, the number of paths that go through the forbidden point C. These bad paths go from O to C to B to A; therefore, their number is $\binom{4}{1} \cdot \binom{4}{3} \cdot \binom{4}{2} = 96$. Consequently, the number of good paths is $420 - 96 = 324$.
13. The left-hand side is the number of all subsets of $[n]$. Indeed, when we select a subset of $[n]$, we make n decisions, and in each of these decisions, we have two choices. That is, for each element of i $in[n]$, we decide whether we put i into our subset or not.

    The right-hand side is the same, counted first by the *size* of the subsets. Indeed, $\binom{n}{k}$ is the number of all k-element subsets of $[n]$.

    Alternatively, 2^n is the number of all northeastern lattice paths starting at $(0, 0)$ and having n steps (ending on the diagonal $x + y = n$). On the right-hand side, we count these same paths, according to their endpoints $(k, n - k)$.
14. The left-hand side is just the number of ways to go from $(0, 0)$ to $(k, n - k)$ by a northeastern lattice path. The right-hand side is the

same, counted by when the *last east step is taken*. As there are altogether k east steps, the last east step has to be preceded by $k-1$ east steps and i north steps, where $0 \le i \le n-k$. In other words, i describes how late the last east step is taken (it is the $i+k$th step). The number of ways the i north steps and $k-1$ east steps preceding the last east step could be taken is $\binom{i+k-1}{k-1}$. Summing over all possible values of i, we get the right-hand side.

15. The left-hand side is the number of ways to choose two subsets of $[n]$, one of which is of size p, and the other one of which is of size q. The right-hand side is the same, counted by the size of the *intersection* of these two subsets.

16. We know from Supplementary Exercise 7 that the total number of subsets of $[n]$ that are of *even* size is

$$A = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} = 2^{n-1}.$$

As $n = 4k+2$, we see that, for even m, exactly one of m and $n-m$ is divisible by four, so exactly one of the equal numbers $\binom{n}{m}$ and $\binom{n}{n-m}$ is included in the sum

$$B = \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots + \binom{n}{n}.$$

Therefore, $A = 2B$, proving that $B = 2^{n-2}$.

17. That expression is equal to 3^n, as can be seen from the binomial theorem, setting $x = 4$ and $y = -1$.

18. As the position of the Lakers and the Sonics are known, we only have to consider the positions of the remaining five teams. Let us replace each of the Kings, the Trailblazers, and the Clippers with a symbol X. These three symbols X and the two remaining teams can be arranged in $\binom{5}{3,1,1} = 5!/3! = 20$ ways. Each such arrangement corresponds to a valid standing, since the first X can be switched back to the Kings, the second X to the Trailblazers, and the third X to the Clippers. So there are ten valid standings.

19. There are 720 solutions. Glue the Rockets and the Grizzlies together. There are $6! = 720$ ways to permute the remaining six symbols. Half of these are wrong, since the Nuggets must be behind the Spurs. On the other hand, the Rockets and the Grizzlies can be unglued in two different ways, multiplying the number of good arrangements by two.

20. Let us first choose the five teams that the Magic will play twice at home and once away. The number of ways to do this is $\binom{9}{5}$.

Once this is done, we can proceed similarly to Example 1.43, except that we now need different symbols to represent home and away games against the same team. Therefore, no symbol occurs more than twice, and the number of symbols that occur twice is $4 \cdot 2 + 9 = 17$. Therefore, the total number of possible schedules is

$$\binom{9}{5} \cdot \frac{71!}{2^{17}}.$$

21. There are 2^n choices for S, there are 2^n choices for T, so there are $2^n \cdot 2^n = 4^n$ choices for (S, T).

22. (a) This number is 3^n. Indeed, each element of $[n]$ can be in T, or in S but not in T, or not in S.
    (b) This number is 3^n again. Indeed, R and U are disjoint if and only if the complement S of R contains U. The result then follows from part (a).

23. Let us ask the tennis players to form a line. This line can be formed in $(2n)!$ ways; let all those possible lines form the set T. Then, let us ask the first person in the line to play with the second, the third person in the line to play with the fourth, and so on. Let S be the set of pairings that are formed this way. We claim that each element of S corresponds to $n!2^n$ different elements of T. Indeed, if we permute the *pairs* of a line in T, the corresponding pairing in S will not change, which explains the $n!$ factor in our claim. Moreover, if we switch the $2i-1$st and the $2i$th person of a line in T, the corresponding pairing in S will not change, which explains the 2^n factor in our claim. Therefore, by the division principle, there are

$$|S| = \frac{|T|}{d} = \frac{(2n)!}{n! \cdot 2^n}$$

ways to match the tennis players.

24. Let $x_n = \frac{n!}{(n/r)^n}$. Then

$$R_n = \frac{x_{n+1}}{x_n} = r \cdot \left(\frac{n}{n+1}\right)^n = r a_n. \tag{1.14}$$

It follows from the hint given in the exercise that each element of the sequence a_n is larger than $1/e$. Since we know that $e < r$, this means that $R_n > 1$ always, that is, the sequence $n!$ grows faster than the sequence $(n/r)^n$. Since $1! > 1/r$, the result follows.

25. (a) As we have to take $2n$ steps, n of which have to be to the east, $a_n = \binom{2n}{n}$.
    (b) Let p be a path enumerated by b_n. Then p must touch the line $y = x + 1$; let X be the first point where this happens. Our

bijection f will take the initial part of p that is between $(0,0)$ and X and reflect it through the line $y = x + 1$. This will turn p into a path $f(p) \in T$. It is easy to see that f is a bijection, since each path $t \in T$ must intersect the $y = x + 1$ line and, therefore, the point X can always be recovered. See Figure 1.12 for an example of our bijection.

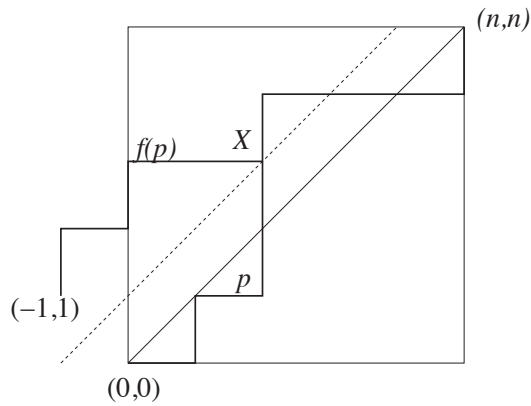

Figure 1.12
The bijection f of Exercise 25.

(c) As we have to take $2n$ steps, $n + 1$ of which have to be to the east, $|T| = \binom{2n}{n+1} = \binom{2n}{n-1}$. Therefore, part (b) implies that $|S| = b_n = \binom{2n}{n-1}$, so by part (a),

$$c_n = a_n - b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}. \tag{1.15}$$

26. This result was first proved by Bertrand [9] (who only gave the outline of a proof), and André [5], in 1887. The problem is often called the *ballot problem*.

We prove the statement by induction on $a + b$, the initial case of $a + b = 0$ being easy to verify. Note that the statement is also true if $b = 0$, or if $kb = a$, that is, if (a, b) is on either boundary line of the allowed domain.

By the addition principle, the number of lattice paths with the described property from $(0,0)$ to (a,b) is certainly equal to the sum of the numbers of such paths from $(0,0)$ to $(a-1,b)$ and from $(0,0)$ to $(a,b-1)$. Therefore, using the induction hypothesis, the number of paths from $(0,0)$ to (a,b) with the described property is $\binom{a-1+b}{a-1}\frac{a-k(b-1)}{a+b-1} + \binom{a+b-1}{a}\frac{a-1-kb}{a+b-1}$. So our statement will be proved

if we can prove the identity

$$\binom{a-1+b}{a-1}\frac{a-1-kb}{a+b-1}+\binom{a+b-1}{a}\frac{a-k(b-1)}{a+b-1}=\binom{a+b}{a}\frac{a-kb}{a+b}.$$

Dividing both sides by $\binom{a+b}{a}$, then multiplying by $(a+b)(a+b-1)$, produces an identity whose proof requires only routine algebra.

27. Expand $(1+x)^n$ by the binomial theorem, then take derivatives, and multiply both sides by x, to get

$$xn(1+x)^{n-1}=\sum_{k=1}\binom{n}{k}kx^k.$$

Set $x=1/n$ to get the formula

$$\left(1+\frac{1}{n}\right)^{n-1}=\sum_{k=1}^{n}\binom{n}{k}\frac{k}{n^k}. \tag{1.16}$$

28. First, using the formula $\binom{n+1}{k+1}=\binom{n}{k}+\binom{n}{k+1}$, we break up our expression on the right-hand side as

$$\sum_{k=1}^{n}\binom{n+1}{k+1}\frac{k}{n^k}=\sum_{k=1}^{n}\binom{n}{k}\frac{k}{n^k}+\sum_{k=1}^{n}\binom{n}{k+1}\frac{k}{n^k}. \tag{1.17}$$

The result of the previous exercise provides a closed formula for the first term of the right-hand side. The second term can be transformed as follows:

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k+1}\frac{k}{n^k} &= n\sum_{k=1}^{n}\binom{n}{k+1}\frac{k}{n^{k+1}}\\ &= n\sum_{k=1}^{n}\binom{n}{k+1}\frac{k+1}{n^{k+1}}-n\sum_{k=1}^{n}\binom{n}{k+1}\frac{1}{n^{k+1}}.\end{aligned}$$

Now note that the sum in the first term of the last line is almost identical to the expression of the previous exercise, and so it is equal to $\left(1+\frac{1}{n}\right)^{n-1}-1$. The sum in the second term of that line equals $\left(1+\frac{1}{n}\right)^{n}-2$ by the binomial theorem, since the terms indexed by 0 and 1 are missing.

This shows that the last term on the right-hand side of (1.16) is equal to

$$n\left[\left(1+\frac{1}{n}\right)^{n-1}-1\right]-n\left[\left(1+\frac{1}{n}\right)^{n}-2\right]=n-\left(1+\frac{1}{n}\right)^{n-1}.$$

The proof is now immediate, since we have computed in the previous exercise that the first term of the left-hand side of (1.16) is precisely $\left(1+\frac{1}{n}\right)^{n-1}$.

29. No matter what the organizers do, it will always take 84 games to determine a winner. Indeed, 84 people need to be eliminated, and that takes 84 games.

30. The possible numbers for victories are $0, 1, 2, \cdots, 15$. This is a list of 16 numbers, so each of them must indeed occur as the number of victories of a player. Therefore, the winner won 15 games, the runner-up won 14 games, the second runner-up won 13 games, and so on, with the person finishing sixth winning 10 games.

31. The left-hand side counts each element of $\cup_{i=1}^{k} A_i$ exactly once. The right-hand side counts these same elements, but if x occurs in t different A_i, then it is counted t times. As $t \geq 1$ for all elements of $\cup_{i=1}^{k} A_i$, the statement follows.

32. The proof is very similar to that of the pigeonhole principle. Simply, (1.11) has to be replaced by the chain of inequalities

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \leq |A_1| + |A_2| + \cdots + |A_n| \leq kr.$$

The previous exercise shows that this chain of inequalities indeed holds. The rest of the proof is unchanged.

33. We claim that it suffices to consider points whose first coordinate is 1, 2, 3, or 4 and whose second coordinate is at least one and at most 82. In other words, we consider points in a rectangular grid that has 82 rows of length 4 each.

As each row has four points, there are $3^4 = 81$ possible colorings for each row. Since there are 82 rows, there will be two rows whose colorings are identical. Finally, at least one color has to be repeated in one (and therefore, the other) of these rows since we can only use three colors to paint four points. Therefore, there will be four points of the same color, and they will form a rectangle.

34. The possible prime divisors of our integers are 2, 3, 5, and 7. So they are all of the form $2^a 3^b 5^c 7^d$, where the exponents are integers. As far as divisibility by 3 is concerned, each exponent can be of the form $3k$, $3k+1$, or $3k+2$. So (a, b, c, d) can be of $3^4 = 81$ different types. Therefore, by the pigeonhole principle, there are three integers among our 175 integers for which the type of (a, b, c, d) is the same. The product of these three integers will be a cube since it will contain each of 2, 3, 5, and 7 raised to an exponent that is divisible by three.