Definition 10.1 *A* magic square *is a square matrix with nonnegative integer entries in which all row sums and column sums are equal.*

Other definitions of magic squares exist. For instance, we could require that diagonals also have the same sum as rows and columns, or we could require that no rows or columns contain the same number twice. In this book, however, we will only use Definition 10.1.

Note that, in a magic square, it cannot happen that all row sums are equal to a, and all column sums are equal to b, while $a \neq b$. Indeed, that would mean that the sum of all elements in the magic square is equal to *na when counted by rows* and to *nb when counted by columns*, which would be a contradiction.

This problem can be generalized in at least two directions. Let us call rows and columns by the same word *lines*, and let $H_n(r)$ be the number of magic squares of size $n \times n$ with line sum r. Our example asked what the value of $H_3(20)$ was. Instead, we could ask what the value of $H_3(r)$ is for any given line sum r, or we could ask what the value of $H_n(60)$ is for any side length n. In other words, we could keep the size of the magic square fixed and study how $H_n(r)$ changes in function of r, or we could keep the line sum r fixed and see how $H_n(r)$ changes in function of n.

Quick Check

1. Compute $H_3(1)$ and $H_3(2)$.
2. The solution of the previous Quick Check exercise shows that

$$H_3(2) = \frac{(H_3(1) + 1) \cdot H_3(1)}{2}. \tag{10.1}$$

   Explain what causes this equality to hold.
3. If we increase the size of our magic squares from 3 to 4, then the equality analogous to (10.1) will not hold. Explain why.

10.2 Magic squares of fixed size

Let us start with small fixed values of n. If $n = 1$, then there is only one way to construct a magic square of size $n \times n$ having line sum r, namely, by setting its only entry to r. Therefore, $H_n(r) = 1$.

This was not terribly difficult. The case of $n = 2$ is not very complicated either. Indeed, if we know the top left element of a 2×2 magic square with line sum r, we can compute all its elements, as shown in Figure 10.2.

Because x and $r - x$ must be nonnegative integers, we must have $0 \leq x \leq r$, which leaves $r + 1$ possibilities for x. Since x completely determines the magic square, $H_2(r) = r + 1$ follows.