example of $n = 3$, and $K = \{1\}$. Then, by the same argument as above for $|B_i(r) \cup B_j(r)|$, we have

$$|B_{i_1}(r) \cap \cdots \cap B_{i_z}(r)| = \binom{r - |f_K| + m - 1}{m - 1},$$

which is again a polynomial of r. Therefore, by the inclusion-exclusion principle, we obtain

$$|B_1(r) \cup B_2(r) \cup \cdots \cup B_s(r)| = (-1)^{k-1} \sum_{\substack{|K|=k \\ K \subseteq [s]}} | \cap_{i_j \in K} B_{i_j}(r)|.$$

Therefore, $|B_1(r) \cup B_2(r) \cup \cdots \cup B_s(r)|$ is a sum of several polynomials of r, so it is a polynomial r. Thus the number of all surplus decompositions is a polynomial function of r, and since the number $\binom{n!+r-1}{n!-1}$ of all decompositions is also a polynomial function of r, our theorem is proved by the subtraction principle. ◊

Quick Check

1. Find the number of 3×3 magic squares with line sum r that are symmetric to both diagonals.
2. Find the number of 3×3 magic squares with line sum r that have a vertical line of symmetry.
3. Find the number of 3×3 magic squares with line sum r that have both a vertical and a horizontal line of symmetry.

10.3 Magic squares of fixed line sum

Let us now assume that our magic squares have a fixed line sum r and that their side length n is changing. In other words, we are looking at $H_n(r)$ as a function of n, with r fixed. If you prefer the distribution example, more and more children will play, and more and more toy types will be available, but the number of toys each child gets will stay the same, and so will the number of toys of each type.

What can be said for small values of r? If $r = 0$, then $H_n(0) = 1$ for all n, since all entries of the magic square have to equal zero in this case. If $r = 1$, then $H_n(1) = n!$, as we have seen in Proposition 10.5. So, even in this simple case, $H_n(r)$ is *not* a polynomial function of n, but a much faster growing function.

The task of finding a formula for $H_n(2)$ is significantly more difficult. The key element of our proof is the following lemma, which is due to Békéssy.

Lemma 10.16 *Let $T_n(2)$ be the number of $n \times n$ magic squares with line sum 2 which do not contain an entry equal to 2. Then, for all $n \geq 2$,*

$$T_n(2) = \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} n!(2n-2k-1)!!}{2^n}. \qquad (10.15)$$

Proof: Recall the distribution problem at the very beginning of this chapter. In the language of that problem, our problem can be expressed as follows: We have $2n$ blocks, two of them red, two of them blue, and so on, two of each of n colors. Blocks of the same color are identical. In how many ways can we distribute these $2n$ blocks to n children so that each child gets two blocks *and no child gets two blocks of the same color?*

First, consider the slightly different problem in which even blocks of the same color are different (for example, they are numbered 1 and 2). If the number of all distributions is $T'_n(2)$, then we have $T'_n(2) = 2^n T_n(2)$. This is because, if the two blocks of color i are different, we can swap them and get a new distribution. (Note that here we use the fact that no child has two blocks of the same color.) So we might as well try to find $T'_n(2)$ instead of $T_n(2)$.

We will find $T'_n(2)$ by an inclusion-exclusion argument. Let A be the set of distributions of the $2n$ blocks to the n children so that each child gets two blocks which may or may not be the same color. Then

$$|A| = (2n)!/2^n = (2n-1)!!n!. \qquad (10.16)$$

Indeed, we can just line up the $2n$ blocks, then give the first two to the first child, the third and fourth to the second child, and so on. There are $(2n)!$ ways to do this. Each distribution will be counted 2^n times, however, since swapping blocks in positions $(2i-1)$ and $2i$ of the line does not result in a new distribution.

Now let us count how many of these $|A|$ distributions are "bad," that is, give two blocks of the same color to some children. Let A_i be the set of distributions in which child i gets two blocks of the same color. Then

$$|A_{i_1} \cap A_{i_2} \cap \ldots A_{i_k}| = (n)_k (2n-2k)!/2^{n-k} = n!(2n-2k-1)!!.$$

Since we first can choose the k colors of which children $i_1, i_2, \cdots, i_k$ will get their blocks in $(n)_k$ ways, then we can distribute the remaining $2(n-k)$ blocks arbitrarily in $(2n-2k-1)!!(n-k)!$ ways, as explained in the proof of (10.16). Note we use the fact that $(n)_k(n-k)! = n!$.

Using the inclusion-exclusion principle, we get that

$$\begin{aligned} T'_n(2) &= |A| - |A_1 \cup A_2 \cup \cdots A_n| \\ &= (2n-1)!!n! - \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} n!(2n-2k-1)!! \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k n!(2n-2k-1)!!. \end{aligned}$$

Applying the identity $T_n'(2) = 2^n T_n(2)$, our statement is proved. ♢

The following corollary probably looks a little surprising at first sight.

Corollary 10.17 *Let* $T(x) = \sum_{n=0}^{\infty} T_n(2)\frac{x^n}{n!^2}$*, where we set* $T_0(2) = 1$*. Then*

$$T(x) = \frac{e^{-x/2}}{\sqrt{1-x}}. \tag{10.17}$$

Recall from Section 3.5 that, for a sequence of real numbers a_n, the power series $\sum_{n\geq 0} a_n \frac{x^n}{n!^2}$ is called the *doubly exponential generating function* of the sequence a_n. So $T(x)$ is the doubly exponential generating function of the sequence $T_n(2)$, and Corollary 10.17 shows that it has a fairly compact form.

Proof: (of Corollary 10.17) We will show that the two power series on the two sides of (10.17) are equal by showing that, for all n, the coefficients of $x^n/(n!)^2$ are the same in both power series.

On the left-hand side, this coefficient is equal to

$$\frac{\sum_{k=0}^{n}(-1)^k\binom{n}{k}n!(2n-2k-1)!!}{2^n}$$

by Lemma 10.16. The right-hand side is the product of two power series, $e^{-x/2}$ and $(1-x)^{-1/2}$. The expansion of $e^{-x/2}$ is straightforward, namely,

$$e^{-x/2} = \sum_{k\geq 0}(-1)^k\frac{x^k}{2^k k!}.$$

The expansion of $(1-x)^{-1/2}$ can be obtained by the binomial theorem. Note that

$$\binom{-1/2}{i} = \frac{(-1/2)(-3/2)\cdots((-2i+1)/2)}{i!} = \frac{(-1)^i(2i-1)!!}{2^i i!}.$$

Therefore, by the binomial theorem,

$$(1-x)^{-1/2} = \sum_{i\geq 0}\binom{-1/2}{i}(-x)^i = \sum_{i\geq 0}\frac{(2i-1)!!}{2^i i!}x^i. \tag{10.18}$$

Multiplying the expansions of $e^{-x/2}$ and $(1-x)^{-1/2}$ together, we get that the right-hand side of (10.17) is equal to

$$\sum_{n\geq 0} x^n \sum_{k\geq 0}^{n}\frac{(-1)^k}{2^k k!}\cdot\frac{(2n-2k-1)!!}{(n-k)!2^{n-k}} = \sum_{n\geq 0} x^n \sum_{k=0}^{n}\frac{(-1)^k(2n-2k-1)!!}{k!(n-k)!\cdot 2^n}.$$

The coefficient of x^n on the right-hand side of (10.17) is

$$\sum_{k=0}^{n}\frac{(-1)^k(2n-2k-1)!!}{k!(n-k)!\cdot 2^n},$$

implying that the coefficient of $x^n/(n!)^2$ is

$$(n!)^2 \sum_{k=0}^{n} \frac{(-1)^k(2n-2k-1)!!}{k!(n-k)!2^n} = \frac{n!}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k-1)!!,$$

which indeed agrees with the corresponding coefficient on the left-hand side. This proves our corollary. ♢

That's fine, you might say, but how did you know in advance that $\frac{e^{-x/2}}{\sqrt{1-x}}$ was the right expression for $T(x)$? This is a very appropriate question. That is, the "problem" with the above proof is that the result was *verified* rather than *deduced*. A proof where the answer is actually *deduced* can be found by solving Exercises 11 and 12.

The following lemma shows the very close connection between the numbers $T_n(2)$ and $H_n(2)$.

Lemma 10.18 *Let* $H(x) = \sum_{n\geq 0} H_n(2)\frac{x^n}{n!^2}$ *be the doubly exponential generating function of the sequence* $H_n(2)$*. Then*

$$H(x) = \frac{e^{x/2}}{\sqrt{1-x}}. \tag{10.19}$$

Proof: Note that by Corollary 10.17, our lemma is equivalent to

$$H(x) = e^x T(x).$$

We will prove the latter by showing that the coefficients of $x^n/(n!)^2$ agree on both sides, for all n.

On the left-hand side, this coefficient is $H_n(2)$. Let us expand the right-hand side as

$$\begin{aligned} e^x T(x) &= \left(\sum_{k\geq 0} \frac{x^k}{k!}\right) \cdot \left(\sum_{i\geq 0} T_i(2)\frac{x^i}{i!^2}\right) \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{T_{n-k}(2)}{(n-k)!^2} x^n. \end{aligned}$$

Therefore, the coefficient of x^n on the right-hand side is $\sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{T_{n-k}(2)}{(n-k)!^2}$, and so that of $x^n/(n!)^2$ is

$$n!^2 \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{T_{n-k}(2)}{(n-k)!^2} = \sum_{k=0}^{n} \binom{n}{k}^2 k! \cdot T_{n-k}(2). \tag{10.20}$$

Consequently, our lemma will be proved if we can show that the right-hand side of this last equation (10.20) is equal to $H_n(2)$. This can be done as follows:

Let us assume that H is a magic square counted by $H_n(2)$ that contains exactly k entries that are equal to 2. There are $\binom{n}{k}^2 k!$ ways to choose the location of these k entries. Then we must fill out the remaining $(n-k) \times (n-k)$ grid so that each row and column contains two entries equal to 1. This can be done in $T_{n-k}(2)$ ways. This proves that (10.20) holds, completing the proof of the lemma. ♢

Finally, we are in a position to state and prove our formula for the numbers $H_n(2)$.

Theorem 10.19 *For all positive integers $n \geq 1$,*

$$H_n(2) = \frac{n!}{2^n} \sum_{k=0}^{n} \binom{n}{k} n!(2n-2k-1)!!. \tag{10.21}$$

Proof: By Lemma 10.18, $H_n(2)$ is the coefficient of $x^n/(n!)^2$ in $H(x) = e^{x/2}/\sqrt{1-x}$. On one hand, we have

$$e^{x/2} = \sum_{k \geq 0} \frac{x^k}{2^k k!}.$$

On the other hand, we have recently computed $(1-x)^{-1/2}$ in (10.18). Using that formula, we get that

$$H(x) = \sum_{n \geq 0} x^n \sum_{k=0}^{n} \frac{(2n-2k-1)!!}{k!(n-k)! \cdot 2^n}.$$

Note that the computation is the same as in the proof of Corollary 10.17, but with the $(-1)^k$ omitted. This shows that the coefficient $H_n(2)$ of $x^n/(n!)^2$ in $H(x)$ is $\frac{\sum_{k=0}^{n} \binom{n}{k} n!(2n-2k-1)!!}{2^n}$ as claimed. ♢

A couple of questions are in order. Is there a reasonably simple proof of this formula that does not use generating functions? If yes, does that combinatorial proof explain the 2^n term in the denominator? Note that it is certainly not obvious why $n! \sum_{k=0}^{n} \binom{n}{k} n!(2n-2k-1)!!$ should be divisible by 2^n.

The answer to both of these questions is in the affirmative, as shown by a nice argument by W. Griffiths [40]. The reader is invited to consult Exercises 4 and 5, which walk the reader through that interesting proof.

Quick Check

1. How many $n \times n$ magic squares are there with line sum 2 that contain $n-2$ digits 2?