2. How many $n \times n$ magic squares are there with line sum 2 that contain $n-3$ digits 2?
3. How many $n \times n$ magic squares are there with line sum 3 that contain $n-3$ digits 3?

10.4 Why magic cubes are different

Now consider the following enhanced version of our original distribution problem. Again, we want to distribute 60 building blocks to three children so that each child gets 20 blocks. 20 blocks are red, 20 are blue, and 20 are green, and blocks of the same color are identical. This is where the original problem ended. Assume, however, that the children come back to play tomorrow, and the day after tomorrow. On each of these three days, we distribute the blocks following the above rule (each child gets 20), however, we are also looking for a long-term balance. Therefore, we require, that at the end of the three-day period, the number of red blocks each child had throughout the entire period be 20, the number of blue blocks each child had throughout this period be 20, and the number of green blocks each child had throughout this period be 20. In how many ways can all this be done?

We know from Section 10.1 that an acceptable distribution of blocks can be represented by a magic square of side length 3 and line sum 20, and therefore the number of acceptable distributions for any given day is $H_3(20)$. However, we cannot just take three magic squares, put them on top of each other, and say that the obtained three-dimensional array of size $3 \times 3 \times 3$ represents a solution to the entire problem. This is because of the extra requirement on the balance of colors. That is, we are looking for distributions so that, if we put the corresponding three magic squares on top of each other, the sum of the entries in each *vertical line* is also 20. One possible distribution is shown in Figure 10.7. Let A, B, and C denote the three children.

	A	*B*	*C*
red	5	6	9
blue	6	7	7
green	9	7	4

today

	A	*B*	*C*
red	8	6	6
blue	7	5	8
green	5	9	6

tomorrow

	A	*B*	*C*
red	7	8	5
blue	7	8	5
green	6	4	10

day after tomorrow

Figure 10.7
A possible three-day distribution.

Such a construction—an $n \times n \times n$ array of nonnegative integers in which the sum of each line, that is, (row, column, and vertical line), is the same—is called a *magic cube*. So the array in Figure 10.7 is an example of a magic cube of side length 3 and line sum 20.

Let $C_n(r)$ be the number of magic cubes of side length n and line sum r. The reader should check that $C_1(r) = 1$ and $C_2(r) = r + 1$. After this, the reader might be thinking that the enumeration of magic cubes is in fact very similar to that of magic squares. This is not true, however, as the smallest nontrivial case, that of $n = 3$, shows. A $2\times2\times2$ subcube completely determines a $3 \times 3 \times 3$ magic cube, so $C_3(r) \le r^8$. Therefore, if $C_3(r)$ is a polynomial, its degree can be at most eight. It is not difficult (see Exercise 20) to write a computer program that computes the first 10 values of $C_3(r)$. These values are shown in Figure 10.8.

r	$C_3(r)$
0	1
1	12
2	132
3	847
4	3921
5	14286
6	43687
7	116757
8	280656
9	619219

Figure 10.8
The values of $C_3(r)$ for $0 \le r \le 9$.

We will now explain how to deduce from those 10 values that $C_3(r)$ is *not* a polynomial.

Let $f : \mathbf{N} \to \mathbf{R}$ be any function, and let $\Delta(f)(n) = f(n+1) - f(n)$. Similarly, $\Delta^2(f)(n) = \Delta(\Delta(f)(n)) = (f(n+2) - f(n+1)) - (f(n+1) - f(n))$, and one can obtain Δ^d along the same lines. The following proposition characterizes polynomials of degree at most d in terms of Δ.

Proposition 10.20 *Let $p : \mathbf{N} \to \mathbf{R}$ be a polynomial. Then the degree of p is at most d if and only if $\Delta^d(p)$ is a constant function.*

Proof: We prove the statement by induction on d. For $d = 0$, the claim trivially holds. Now assume the claim holds for d and prove it for $d + 1$.

(a) (the "only if" part) Let p be a polynomial of degree $d + 1$. Then $p(n) = a_{d+1}n^{d+1} + a_d n^d + \cdots + a_1 n + a_0$. Therefore, $p(n + 1) = a_{d+1}(n+1)^{d+1} + a_d(n+1)^d + \cdots + a_1(n+1) + a_0$. From this, we see that n^{d+1} cancels out in $p(n+1) - p(n) = \Delta(p)(n)$, so $\Delta(p)$ is a polynomial of degree at most d. Therefore, by the induction hypothesis, $\Delta^d(\Delta(p))$ is constant, in other words, $\Delta^{d+1}(p)$ is constant.

(b) (the "if" part) Let p be a function so that $\Delta^{d+1}(p)$ is constant. Then $\Delta^d(\Delta(p))$ is constant, so, by the induction hypothesis, $\Delta(p)$ must be a polynomial of degree at most d. This implies that p is of degree at most $d+1$. Indeed, if we have $p(n) = \sum_{i=0}^{h} a_i n^i$, then we also have $\Delta(p)(n) = \sum_{i=0}^{h} a_i(n+1)^i - \sum_{i=0}^{h} a_i n^i$, which contains a term of degree $h-1$.

$\Diamond$

We point out that Proposition 10.20 remains true if we change its wording a little bit. Instead of requiring that p is a polynomial, we could simply require that p be a function that satisfies the requirement that $\Delta^d(p)$ is constant. Then the above argument would show that p is a polynomial of degree at most d. However, we will not need this stronger form.

Using our numerical data, we can easily compute that $\Delta^8(C_3(r))(0) \neq \Delta^8(C_3(r))(1)$; therefore, $C_3(r)$ cannot be a polynomial of degree at most 8, so $C_3(r)$ cannot be a polynomial at all.

Hopefully, you are wondering now what could possibly cause this unexpected behavior of the function $C_3(r)$. You might also be wondering if $C_3(r)$ is just one exception or if maybe $C_n(r)$ is never a polynomial if $n \geq 3$.

Let us try to copy the proof of Theorem 10.3, the theorem showing that $H_n(r)$ is always a polynomial. The point where we get stuck is Lemma 10.6. That lemma has no three-dimensional analog. In fact, that lemma does not hold for magic cubes, even if we restrict our attention to magic cubes of side length 3 and line sum 2. Figure 10.9 shows a counterexample.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Figure 10.9
An irreducible magic cube of line sum 2.

Indeed, the first level can be decomposed in only one way, and its bottom right corner makes any decompositions of the other two levels impossible to fit in a magic cube.

Let us call a magic cube *irreducible* if it cannot be decomposed into the sum of magic cubes of smaller line sums. The above example then shows that not all irreducible magic cubes have line sum 1. It can be shown, however, that all irreducible magic cubes of side length 3 have line sum either 1 or 2. Then it can be proved, in a manner similar to the proof of Theorem 10.3, that both $C_3(2r)$ and $C_3(2r+1)$ are polynomials. Now that we know that there are irreducible magic cubes of side length 3 with line sum 2, this result is not