10.7 Exercises

1. Prove our claim made in part (a) of the proof of Theorem 10.2 stating that, the 4-tuples of nonnegative integers (a, b, c, d) satisfying (10.8) and the 4-tuples $(a, 2a + d - c, a + b + d - c, b + d)$ satisfying (10.8) are in bijection.
2. Prove our claim made in part (c) of the proof of Theorem 10.2 stating that if $a > c$ and $b > c$, then inequalities (10.3)–(10.6) are all redundant.
3. Assume we know that p is a polynomial of degree m, and also assume we know the values of $p(0), p(1), \cdots, p(m)$. How can we find p? (Define a general strategy that shows that p can be found. Do not discard a method simply because it may require a lot of computation.)
4. Let A be a $2n \times 2n$ matrix with nonnegative integer entries. For any $i \in [n]$, let us say that rows $2i - 1$ and $2i$ form a *row pair*, and columns $2i-1$ and $2i$ form a *column pair*. We say that A is a *double magic square* of order n if it satisfies the following criteria:

   (a) The sum of entries of each row pair and each column pair is 2,
   (b) if a row pair contains a 2, then that 2 must be in the *top* row of the row pair, and
   (c) there is at most one positive integer in any row or column.

   See Figure 10.10 for an example. Prove that the number $DM(n)$ of double magic squares of order n is

$$\sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} 2^k (2n - 2k)!.$$

0	0	2	0
0	0	0	0
0	1	0	0
1	0	0	0

Figure 10.10
A double magic square of order 2.

5. (a) Prove that there exists a 2^{2n}-to-one map f from the set S of all double magic squares of order n onto the set T of all magic squares of size $n \times n$ having line sum 2.
   (b) Deduce Theorem 10.19.
6. Let $P_3(r)$ be the number of 3×3 magic squares that are symmetric to their main diagonal and have line sum r. Is $P_3(r)$ a polynomial?
7. Prove that $P_{n+1}(1) = P_n(1) + nP_{n-1}(1)$.
8. Find the exponential generating function of the numbers $P_n(1)$.
9. Let P be a magic square that is symmetric to its main diagonal. Is it true that P is the sum of some *symmetric* permutation matrices?
10. Show an example of three disjoint sets of permutation matrices having the same sum.
11. Prove (preferably with a direct argument) the recurrence relation

$$T_{n+1}(2) = n^2(n+1)T_{n-1}(2)/2 + n(n+1)T_n(2).$$

12. Use the result of the previous exercise to find the doubly exponential generating function $T(x) = \sum_{n \geq 0} \frac{T_n(2)}{(n!)^2} x^n$.
13. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k(2) T_{n-k}(2) = n!^2.$$

14. Prove (by any method) that the recurrence relation

$$H_n(2) = n^2 H_{n-1}(2) - \binom{n}{2}(n-1)H_{n-2}(2)$$

holds for $n \geq 2$.

15. Compute $C_3(1)$.
16. A *Latin square* is an $n \times n$ square grid that has been filled out by n letters so that each letter occurs in each row and each column exactly once. Figure 10.11 shows a 3×3 Latin square.

A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

Figure 10.11
A 3×3 Latin square.