Construct a bijection from the set of $n \times n$ Latin squares onto that of magic cubes enumerated by $C_n(1)$.

17. Prove that the only irreducible $3 \times 3 \times 3$ magic cubes of line sum 2 are those that can be obtained from our example shown in Figure 10.9 by permuting lines.
18. Compute $C_3(2)$.
19. Prove that $C_n(1)$ is always divisible by $n!$.
20. Write a computer program in any language that computes the value of $C_3(r)$.
21. Let us call a set $A \subset \mathbf{N}^m$ an *antichain* in $\mathbf{N}^m$ if there are no two points $(a_1, a_2, \cdots, a_m)$ and $(b_1, b_2, \cdots, b_m)$ in A so that $a_i \leq b_i$ holds for all $i \in [m]$.

    Prove that all antichains in $\mathbf{N}^m$ are finite.

10.8 Solutions to exercises

1. Given (a, b, c, d), we can compute $\phi(a, b, c, d) = (a, 2a+d-c, a+b+d-c, b+d)$. On the other hand, given $(e, f, g, h) = (a, 2a+d-c, a+b+d-c, b+d)$, we can compute (a, b, c, d) by $a = e$, $b = e - f + g$, $c = e - g + h$, and $d = (-e + f - g + h)$. Thus ϕ has an inverse, and therefore ϕ is a bijection.
2. Inequalities (10.5) and (10.6) are both redundant since d is nonnegative and c is smaller than a and b. Inequality (10.3) is redundant since it follows from (10.7). Indeed,

$$r - a - d \geq r - a - d - (b - c) \geq 0$$

   because $b > c$. Finally, (10.4) is also redundant since it also follows from (10.7). Indeed,

$$r - b - d \geq r - b - d - (a - c) \geq 0$$

   because $a > c$.
3. Let $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots a_1 x + a_0$. Substituting the values $0, 1, \cdots, m$ for x, we get a system of $m+1$ linear equations with the $m+1$ unknowns $a_m, a_{m-1}, \cdots, a_0$. We can then use linear algebra to solve this system.
4. The term indexed by k is the number of double magic squares of order n that contain exactly k entries 2. In these double magic squares, there are $\binom{n}{k}$ ways to choose the row pairs in which those

2s will be put, then there are $\frac{n!}{(n-k)!}$ ways to choose the column pairs. As the 2s can be in either column but must be in the top row of each pair, there are 2^k ways to find their exact positions once the pairs are selected. Deleting the row pairs and column pairs containing a 2, we get one of $(2n-2k)!$ permutation matrices.

5. (a) The definition of f is simplicity itself. Given $A \in S$, let $f(A)$ be the magic square whose (i,j)-element is the sum of the four elements in the intersection of the ith row pair and the jth column pair of A. So if A is the double magic square shown in Figure 10.10, then $f(A)$ is the magic square shown in Figure 10.12.

0	2
2	0

Figure 10.12
The magic square $f(A)$.

It then follows from the first criterion in the definition of double magic squares that $f(A) \in T$. It remains to be proved that, for each $B \in T$, there are exactly 2^{2n} elements $A \in S$ satisfying $f(A) = B$.

Let $B \in T$. We will find all the preimages of B under f as follows: First, double the columns of B. That is, turn B into an $n \times 2n$ matrix B' in which the old column j becomes the new column pair $(2j-1, 2j)$, and then the old (i,j)-entry will give us instruction on what the new entries in positions $(i, 2j-1)$ and $(i, 2j)$ are. Namely, if column j had two entries 1, in positions (i,j) and (i',j), then the corresponding column pair gets an entry 1 in one of its columns, in one of the two positions $(i, 2j-1)$ and $(i, 2j'-1)$, as well as in the *opposite* position of $(i, 2j'-1)$ and $(i, 2j')$. So each column without a 2 in it will let us choose one of two possibilities.

If the (i,j)-entry of B was 2, then again, we have two possibilities. We can set either the $(i, 2j-1)$-entry or the $(i, 2j)$-entry of B' to 2. See Figure 10.13 for an example.

Note that each column of B' has at most one nonzero entry, and that the sum of each column pair is 2.

Now let us double the rows of B' according to slightly different rules. Again, if row i of B contained two entries 1, then the corresponding row pair will get two entries 1 in diagonal positions, letting us choose one of two possibilities. In other words, say

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Figure 10.13
The first step in finding the preimages of B.

row i contains a 1 in position j and another in position j'. Then the new matrix B'' will contain a 1 corresponding to the 1 in position (i, j) of B' either in $(2i - 1, j)$ or in $(2i, j)$. Whichever choice we make here, we have to make the *other choice* for the 1 in position (i, j') of B'.

However, an entry 2 can either stay in the top row of the row pair, or it can be broken up into two entries 1 located in a diagonal of the 2×2 block corresponding to the original entry 2. See Figure 10.14 for an example.

2	0	0	0	0	0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	0	0	0	1
0	0	1	0	0	0
0	0	0	0	1	0

0	2	0	0	0	0
0	0	0	0	0	0
0	0	0	1	0	0
0	0	0	0	0	1
0	0	1	0	0	0
0	0	0	0	0	0

1	0	0	0	0	0
0	1	0	0	0	0
0	0	0	1	0	0
0	0	0	0	0	1
0	0	1	0	0	0
0	0	0	0	1	0

0	1	0	0	0	0
1	0	0	0	0	0
0	0	0	1	0	0
0	0	0	0	0	1
0	0	1	0	0	0
0	0	0	0	1	0

Figure 10.14
Some possibilities for B'' if B' is the matrix at the top of Figure 10.13.

Note that no matter what sequences of choices we made, B'' is a double magic square in S. Furthermore, it follows from our definitions that $f(B'') = B$, since the sum of the entries in the intersection of row pair i and column pair j of B'' is the (i,j)-entry of B. Again by our definitions, all preimages of B under f can be obtained as B'' by this procedure.

Because during this procedure we had 2^{2n} choices to make (2^n connected to the n columns, then 2^n connected to the n rows), the statement is proved.

(b) Apply the division principle and the result of part (a) to the result of the previous exercise.

6. No, it is not. Entries a, b, and c, as shown in Figure 10.15, completely determine such a magic square. So if $P_3(r)$ were a polynomial, its degree could not be larger than three. Computing the first five values of $P_3(r)$, we get $P_3(0) = 1$, $P_3(1) = 4$, $P_3(2) = 11$, $P_3(3) = 23$, and $P_3(4) = 42$. It is then straightforward to check that $\Delta^3(P_3)(0) \neq \Delta^3(P_3)(1)$; therefore, $P_3(n)$ cannot be a polynomial.

a	b	
	c	

Figure 10.15
The entries a, b, and c.

7. Let P be a magic square enumerated by $P_{n+1}(1)$. Then the first row of P contains one 1. This 1 can be either in the first position, and then we have $P_n(1)$ possibilities for the remaining $n \times n$ symmetric magic square, or it is somewhere else. In that case, it can be at n different positions, each of them outside the main diagonal. Whatever position it is in, however, the mirror image of that position must also contain a 1, and then we have $P_{n-1}(1)$ possibilities for the remaining $(n-1) \times (n-1)$ symmetric magic square.

8. Let $P(x) = \sum_{n \geq 0} \frac{P_n(1)}{n!} x^n$. Then multiplying both sides of the recurrence relation of the previous exercise by x^n/n and summing for all $n \geq 0$ leads to the functional equation

$$P'(x) = (1+x)P(x),$$

implying that

$$\frac{P'(x)}{P(x)} = 1 + x,$$

$$\ln P(x) = x + \frac{x^2}{2}.$$

Therefore, $P(x) = e^{x+\frac{x^2}{2}}$. Note that this is equal to the exponential generating function of all involutions of length n. This is not surprising. Indeed, a permutation matrix is symmetric to its main diagonal if and only if it is the permutation matrix of an involution.

9. No, it is not. A counterexample for $n = 3$ is shown in Figure 10.16.

0	1	1
1	0	1
1	1	0

Figure 10.16
A counterexample.

Indeed, this magic square cannot be decomposed into the sum of two symmetric permutation matrices. Since all diagonal entries are zero, a symmetric permutation matrix Q that is part of the decomposition would have to have an even number of 1s in it. That is impossible since Q would have to contain two or four 1s, not three as needed to have line sum 1.

10. We have seen that no such example exists when $n = 3$. An example for $n = 4$ is shown in Figure 10.17.

11. Let b_n denote the number of *almost magic squares*. We say that an $n \times n$ matrix with 0–1 entries is an almost magic square if it contains two entries equal to 1 in each row and column except for the first row and the first column, and it contains one entry equal to 1 in the first row, and one entry equal to 1 in the first column.

    We then claim that

$$T_n(2) = \binom{n}{2}(n-1)b_{n-1}. \tag{10.23}$$

Indeed, take a matrix that is enumerated by $T_n(2)$. There are $\binom{n}{2}$ pairs of positions in which the first column of this matrix can contain

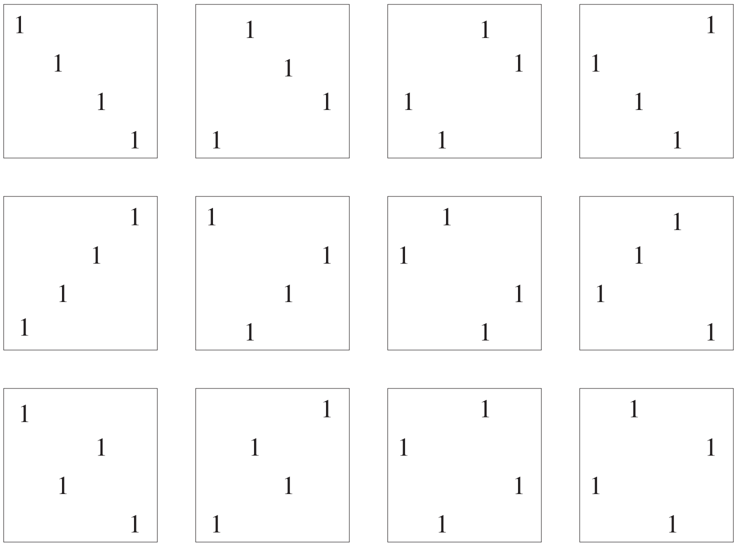

Figure 10.17
All three sets sum to J_4.

its two entries equal to 1. In row R of the first of these two entries, there are $(n-1)$ possible positions for the second entry equal to 1. Finally, if we delete the first column and row R of our matrix, we get an $(n-1)\times(n-1)$ matrix that is an almost magic square, up to a permutation of rows and columns.

On the other hand, we claim that the numbers b_n can also be obtained from the numbers $T_n(2)$. Indeed, we show that

$$b_n = T_{n-1}(2) + (n-1)^2 b_{n-1}. \tag{10.24}$$

Indeed, the two entries equal to 1 in the first row and column of an almost magic square either coincide, in which case the remaining matrix of size $(n-1)\times(n-1)$ is a magic square, or do not coincide. In this case, they can be in $(n-1)^2$ different positions, and, omitting the row and column containing these entries (which are not the first row and not the first column), we get an almost magic square of size $(n-1)\times(n-1)$.

Let us express $T_{n+1}(2)$ by (10.23), then, in the expression thus obtained, express b_n by (10.24) to get

$$T_{n+1}(2) = \binom{n+1}{2} n \cdot b_n = \frac{1}{2}(n+1)n^2 T_{n-1}(2) +$$

$$\frac{1}{2}(n+1)n^2(n-1)^2b_{n-1}.$$

Finally, note that the last term will become simpler if we replace $n(n-1)^2b_{n-1}/2$ by $T_n(2)$, as is allowed by (10.23). This yields

$$T_{n+1}(2) = \binom{n+1}{2} n \cdot T_{n-1}(2) + T_n(2)n(n+1),$$

which was to be proved.

12. Multiply both sides of the recursive relation by $x^n/(n!)^2(n+1)$, then sum for all nonnegative integers n to get

$$\sum_{n\geq 0} \frac{T_{n+1}(2)}{(n!)^2(n+1)} x^n = \sum_{n\geq 0} \frac{\frac{1}{2}\cdot n^2 \cdot T_{n-1}(2)}{(n!)^2} x^n +$$

$$\sum_{n\geq 0} \frac{T_n(2)n}{(n!)^2} x^n.$$

Now note that the previous equation can be written in the shorter form

$$T'(x) = xT(x)/2 + xT'(x).$$

From here, we get

$$\frac{T'(x)}{T(x)} = \frac{x}{2(1-x)}.$$

Let us integrate both sides. On the left-hand side, we currently have $(\ln T(x))'$, so integration leads to $\ln T(x)$. The right-hand side is easy to integrate if we notice that $\frac{x}{2(1-x)} = \frac{1}{2}\sum_{n\geq 1} x^n$. This leads to

$$\ln T(x) = \frac{1}{2}\sum_{n\geq 1} \frac{x^{n+1}}{n+1} = -\frac{x}{2} - \frac{1}{2}\cdot \ln(1-x),$$

and $T(x) = \frac{e^{-x/2}}{\sqrt{1-x}}$ follows by taking exponentials.

13. Taking the product of generating functions $H(x) = \sum_{n\geq 0} H_n(2)\frac{x^n}{n!^2}$ and $T(x) = \sum_{n\geq 0} T_n(2)\frac{x^n}{n!^2}$, and remembering Corollary 10.17 and Lemma 10.18, we get

$$H(x)T(x) = \frac{e^{x/2}}{\sqrt{1-x}} \cdot \frac{e^{-x/2}}{\sqrt{1-x}} = \frac{1}{1-x}. \tag{10.25}$$

On the other hand, expanding the product $T(x)H(x)$, we get

$$H(x)T(x) = \sum_{n\geq 0} x^n \sum_{k=0}^{n} \frac{H_k(2)}{k!^2} \cdot \frac{T_{n-k}(2)}{(n-k)!^2}. \tag{10.26}$$

As the left-hand sides of the last two equations are identical, so too must be their right-hand sides, and in particular, so too must be the coefficients of $x^n/(n!)^2$ in the two right-hand sides. On the right-hand side of (10.25), this coefficient is $n!^2$ since $1/(1-x) = \sum_{n\geq 0} x^n$. On the right-hand side of (10.26), this coefficient is

$$\sum_{k=0}^{n} \frac{n!^2 H_2(k)T_{n-k}(2)}{k!^2 \cdot (n-k)!^2} = \sum_{k=0}^{n} \binom{n}{k}^2 H_2(k)T_{n-k}(2),$$

proving our claim.

14. Note that, defining $a_{-1} = 0$, the recursive formula to be proved also holds for $n = 1$. Because the formula holds for all $n \geq 1$, we can take it for all such n, multiply it by $\frac{nx^{n-1}}{n!^2}$, and sum the obtained equations. We get

$$\sum_{n\geq 1} n \cdot \frac{H_n(2)}{n!^2} x^{n-1} = \sum_{n\geq 1} n \cdot \frac{H_{n-1}(2)}{(n-1)!^2} x^{n-1} - \frac{1}{2} \sum_{n\geq 1} \frac{H_{n-2}(2)}{(n-2)!} x^{n-1}.$$

Note that, by term-by-term comparison, this is equivalent to

$$H'(x) = xH'(x) + H(x) - \frac{1}{2} xH(x),$$

which leads to

$$\frac{H'(x)}{H(x)} = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{1-x},$$

and the solution of this last equation is indeed $H(x)$.

A proof that does not use the explicit formula for $H_n(2)$ can be found in [4].

15. Choose any of the six permutation matrices of size 3×3 for the bottom level. If you choose the matrix π, then there are two matrices that are eligible to be put to the middle and top levels. These are the two matrices that are obtained by multiplying π by the matrices of one of the two fixed point–free permutations of length 3. We can put either of them to the middle level; then we must put the other one to the top level. Therefore, $C_3(1) = 6 \cdot 2 \cdot 1 = 12$.

16. Instead of filling the Latin square with letters, we fill it with numbers, from 1 through n. Let L be such a Latin square. Now we turn all the entries of L into zeros, except for the n entries equal to i, which we turn into ones. This gives us the permutation matrix L_i. Let $f(L)$ be the magic cube whose level i is L_i. It is straightforward to verify that this is a bijection.

17. This can be done by an intelligent proof by cases. If there is a level (or a rotated copy of a level) in a cube that contains three entries

equal to 2, then the cube is not irreducible. There *cannot* be a level with exactly two entries equal to 2. We still have to consider the cases when the highest number m of entries equal to 2 on a level (or rotated copy of it) is 0 or 1. If $m = 0$, then the cube is the difference of the cube full of ones and a cube C of line sum 1, and as such is reducible. (In fact, it is the sum of two circular translates of C.) Finally, if $m = 1$, then we arrive at the example shown in Figure 10.9.

18. Exercise 15 tells us that there are 12 magic cubes of side length 3 and line sum 1. Adding any two of them we get a magic cube enumerated by $C_3(2)$. Moreover, all these two-member sums will be different (why?). This yields $\binom{12+2-1}{2} = \binom{13}{2} = 78$ reducible magic cubes. To count the irreducible ones, consider Figure 10.9. The irreducible magic cube shown there can be transformed into 54 irreducible cubes. Indeed, the sole entry 2 can be placed in 27 different positions, and then the two levels not containing that entry can be permuted in two different ways. The previous exercise then shows that there are no more irreducible cubes either. Therefore, $C_3(2) = 78 + 54 = 132$.

19. **First solution.** Permute the n levels of the magic cube. Each of these $n!$ permutations yields a different magic cube since all n levels are different. Thus, the magic cubes enumerated by $C_n(1)$ can be arranged into subsets, each of which has $n!$ cubes in it.

    Second solution. A magic cube of line sum 1 is just n permutation matrices placed on top of each other. Denote $p_1, p_2, \cdots, p_n$ the permutations corresponding to these matrices. The crucial property of these permutations is that there is no $i \in [n]$ so that $p_j(i) = p_k(i)$ for some $j \neq k$. We claim that this property is preserved if all p_j are multiplied by the same permutation q.

    Indeed, assume that $q(p_j(i)) = q(p_k(i))$ for some $j \neq k$. That is impossible, since the bijection q cannot map the distinct entries $p_j(i)$ and $p_k(i)$ to the same entry. Therefore, for any n-permutation q, the matrices of permutations $qp_1, qp_2, \cdots, qp_n$ also form a magic square. We have thus again arranged our magic cubes into subsets of size $n!$ each.

20. The following is a Maple program computing $C_3(8)$. Changing the value of r in the first line will give other values of $C_3(r)$.

```
r:=8;
i:=0;
for a from 0 to r do
for b from 0 to r do
for c from 0 to r do
for d from 0 to r do
```

```
                              for e from 0 to r do
                              for f from 0 to r do
                              for g from 0 to r do
                              for h from 0 to r do

 if a+b < r+1 and c+d < r+1 and a+c < r+1 and b+d < r+1
 and a+b+c+d > r-1 and  e+f < r+1 and g+h < r+1 and
 e+g < r+1 and f+h < r+1 and e+f+g+h > r-1 and
    a+e < r+1 and b+f < r+1 and c+g < r+1 and d+h < r+1
 and a+b+e+f > r-1 and c+d+g+h > r-1 and a+c+e+g > r-1
and b+d+f+h >  r-1 and  a+b+c+d+e+f+g+h < r+r+r+1 then
                                    i := i+1;
                                 fi;
                              od;
                              od;
                              od;
                              od;
                              od;
                              od;
                              od;
                              od;
                              i;
                              r;
```

21. We prove the statement by induction on m. For $m = 1$, the statement is true, since between two natural numbers there is always one that is at least as large as the other. Now let us assume we know the statement for $m - 1$, and prove it for m.

    Let us assume that $A = \{d_1, d_2, \cdots\}$ is an *infinite* antichain in $\mathbf{N}^m$. For $\mathbf{a} = (a_1, a_2, \cdots, a_m) \in A$, define $f(a) = (a_1, a_2, \cdots, a_{m-1})$. Let $f(A) = \{f(\mathbf{a}) | \mathbf{a} \in A\}$. Then $f(A) \in \mathbf{N}^{m-1}$, so $f(A)$ cannot contain an infinite antichain. The elements of $f(A)$ are pairwise disjoint, in other words, f is an injection (why?).

    Because $f(A)$ is an infinite set with no infinite antichain in it, $f(A)$ must contain an infinite sequence of elements $\mathbf{x_1}, \mathbf{x_2}, \cdots$ so that $\mathbf{x_1} \leq \mathbf{x_2} \leq \cdots$. (This is not obvious; we ask the reader to explain why.) Now let us look at the sequence $S = f^{-1}(\mathbf{x_1}), f^{-1}(\mathbf{x_2}), \cdots$, whose elements are all from A. The only way S would not violate the antichain condition would be if the last coordinates of the elements of S form a strictly decreasing sequence. However, that is clearly impossible, since these coordinates are nonnegative integers.