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Applications of basic methods

2.1 Multisets and compositions

In this section, we are going to prove formulae for the number of ways a positive integer can be written as a sum of nonnegative, or positive, integers. The order of the summands will matter.

2.1.1 Weak compositions

A family of four, Anne, Benny, Charles, and Denise, sat down in their living room to watch a movie. As luck would have it, their home phone rang 10 times during the movie. Furthermore, the answering machine was broken, and the family was expecting an important call, so they answered the phone each time. In order to keep their spirits up, the family agreed that at the end of the movie each person will get a scoop of ice cream for each time that person answered the phone. The 10 scoops of ice cream will certainly be consumed as soon as the movie is over, but how many distributions are possible for those 10 scoops?

Let us start by making two observations. First, the *order* in which the four people pick up the phone does not matter; it is just the number of times each of them answers the phone that matters. This sounds as if we were dealing with a *subset* problem. However, and this is our second observation, no subset of the set $\{A, B, C, D\}$ has 10 elements, so we have to use a different model. We are looking for collections of the letters A , B , C , and D that consist of 10 letters altogether, such as $ABABBBCCDD$.

Such a collection—that is, a collection in which each element has to come from a specific set S , repetitions of elements are allowed, and the order of the elements does not matter—is called a *multiset* over the set S .

In this terminology, our task is to count all 10-element multisets over the four-element set $S = \{A, B, C, D\}$. In order to make this task easier, note that these multisets are totally determined by the *multiplicities* at which each of the four elements occurs in them. That is, if we know that a multiset contains two copies of A , one copy of B , three copies of C , and four copies of D , then we know everything about that multiset. Indeed, there is nothing else to know, since the order of the elements does not matter. This reduces our task

to simply finding the number of ways we can write the integer 10 as a sum of four nonnegative integers (the multiplicities of A , B , C , and D). In other words, we want to find the number of nonnegative integer solutions of the equation $a + b + c + d = 10$. Note that the *order of the summands* matters in this language. Indeed, the solution $2 + 3 + 1 + 4 = 10$ corresponds to a multiset that contains two copies of A , while the solution $1 + 2 + 4 + 3$ corresponds to a multiset that contains one copy of A .

The concept of identifying n -element multisets over a k -element set with ways of obtaining n as a sum of k nonnegative integers leads to an important concept that has its own name.

Definition 2.1 Let a_1, a_2, \dots, a_k be nonnegative integers satisfying

$$\sum_{i=1}^k a_i = n.$$

Then the ordered k -tuple (a_1, a_2, \dots, a_k) is called a weak composition of n into k parts.

The discussion above explains that there is a natural bijection from the set of weak compositions of $[n]$ into k parts and n -element multisets over a k -element set. Therefore, we might as well look for the number of weak compositions of n into k parts instead of the number of n -element multisets over k .

These objects are a little bit more difficult to count than k -element subsets of $[n]$, but the formula enumerating them also consists of a binomial coefficient.

Theorem 2.2 The number of weak compositions of n into k parts is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Proof: Consider n identical balls that have to be distributed into k boxes that are numbered 1 through k . Each such distribution is equivalent to a weak composition of n into k parts, the number of balls in box i being part i .

Now, count the ways of distributing the n balls into the k boxes as follows: Arrange the k boxes in a line so that the numbers on the boxes are increasing order, and boxes i and $i + 1$ are separated by a wall (see [Figure 2.1](#)).



Figure 2.1

Lining up our boxes.