Proof: There is a bijection f from the set W of all weak compositions of $n-k$ into k parts into the set C of all compositions of n into k parts. This bijection simply adds one to each part, assuring that each part will become positive. For example, if $a=(3,0,2)$, then $f(a)=(4,1,3)$. The reader is asked to verify that f is a bijection. Therefore, $|W|=|C|$, and our claim follows from Theorem 2.2. Indeed, replacing n by $n-k$ in the result of that corollary yields

$$|C|=|W|=\binom{n-k+k-1}{k-1}=\binom{n-1}{k-1}.$$

◊

What can we say if *any* number of parts is allowed? The reader is invited to use the result of Corollary 2.5 to find a closed formula (that is, one with no summation signs) for the number of *all* compositions of n. We provide two solutions in Exercise 3. The obtained simple formula may encourage the reader to look for a similar formula for the number of all weak compositions of n into any number of parts, but that is a bird of a different feather. Readers should try to explain why that is, then check their answers in the solution of Exercise 4.

Quick Check

1. Find the number of compositions of n in which the first entry is more than 1.
2. Find the number of weak compositions of 20 into four parts that are identical to their reverses.
3. Find the number of compositions of 11 (into any number of parts) that are identical to their reverses.

2.2 Set partitions

Now we are going to consider a more difficult enumeration problem. Accordingly, the counting formulas that we obtain will be more complicated.

2.2.1 Stirling numbers of the second kind

On a day when we felt particularly brave, we invited some friends of our children to play in our house. Altogether, five children were present. Soon it turned out that no room was large enough for all of them, and therefore they split into the three available rooms, using all three of them. In how many different ways could the children do that?

If you try to answer this question, you soon recognize that it is not asked in a precise way. What is missing is the definition of *different*. That is, do we consider the setup where A and B play in one room, C and D in another room, and E in the remaining room *different* from the setup where E plays in the first room, A and B in the second room, and C and D in the last room?

There are no good and bad answers to this question; both interpretations lead to valid and interesting combinatorial problems. However, the two problems are very closely related. Indeed, it is easy to see that if we consider the three rooms different, the answer will be exactly $3! = 6$ times as much as in the case when the rooms are considered identical. This is because we can permute the rooms in that many ways.

Because of this very close connection between the two problems, it suffices to solve the one in which the rooms are considered identical. That is, it is only the playmates that matter. In other words, two playing arrangements are considered different if there is at least one child whose playmates are not the same in the two arrangements. This situation occurs so often in mathematics that it has its own name.

Definition 2.6 *Let n be a positive integer, and let $k \leq n$ be a positive integer. Let $B = \{B_1, B_2, \cdots, B_k\}$, where $B_i \subseteq [n]$ for all $i \in [k]$, the B_i are nonempty and pairwise disjoint, and $\cup_{i=1}^{k} B_i = [n]$. Then we say that B is a* partition *of $[n]$ into k blocks.*

Example 2.7 *There are six partitions of [4] into three blocks, namely,*

- $\{\{1,2\},\{3\},\{4\}\}$,
- $\{\{1,3\},\{2\},\{4\}\}$,
- $\{\{1,4\},\{2\},\{3\}\}$,
- $\{\{2,3\},\{1\},\{4\}\}$,
- $\{\{2,4\},\{1\},\{3\}\}$, *and*
- $\{\{3,4\},\{1\},\{2\}\}$.

Note that the blocks are sets, so the order of elements within each block does not matter. Also note that in Definition 2.6, B itself is a set, so the order of blocks does not matter either. Therefore, it is no surprise that the number of partitions of [4] into three blocks is six. Indeed, all such partitions will have one doubleton and two singleton blocks, so once we know which elements are in the doubleton block, we know the partition. Since there are $\binom{4}{2} = 6$ ways to choose the doubleton block, our claim follows.

Let us practice this type of counting by answering our original question.

Example 2.8 *The number of partitions of [5] into three blocks is 25.*