Just as for Stirling numbers of the second kind, there are several explicit formulae for the Bell numbers. They are beyond our reach at this point. We can, however, easily prove the following recurrence relation.

Theorem 2.18 *Set* $B(0) = 1$. *Then, for all positive integers* n,

$$B(n+1) = \sum_{k=0}^{n} B(k)\binom{n}{k}.$$

The reader should try to prove this result on his own, following the line of thinking we used in the proof of Theorem 2.16. We provide a proof in the solution to Exercise 15.

Quick Check

1. Find the number of partitions of $[n]$ into two blocks in which 1 and 2 are in different blocks.
2. Prove that for all positive integers $k \leq n$, the inequality $S(n,k) \leq k^n$ holds.
3. Prove that for all positive integers $n \geq 3$, the inequality $B(n) < n!$ holds.

2.3 Partitions of integers

In this section, we return to the study of the number of ways to write a positive integer as the sum of positive integers. However, the parts will have to be in nonincreasing order.

2.3.1 Nonincreasing finite sequences of positive integers

Last time Ms. Smith moved, she had to unload a truck, which involved carrying 20 identical medium-sized boxes into her new house. She wanted to be done quickly and tried to carry several boxes at the same time. However, as time passed, she became more and more tired, so she was happy if she could carry the same number of boxes as on the previous trip from the truck to the house. She could not even think of increasing the number of boxes carried at any given point in time. In how many different ways could she carry the 20 boxes into her new house?

This problem may remind the reader of compositions. Indeed, as the boxes are all identical, we are only concerned with their number. That is, we would like to decompose 20 into a sum of positive integers. However, in contrast to compositions, the *order* of the parts has to be nonincreasing. So $12 + 8$ is acceptable, but $3 + 4 + 1 + 10$ is not. This shows that we are dealing with a different (and we will see, much more difficult) problem than in the previous section. The following definition starts setting the framework for handling this new kind of enumeration problem.

Definition 2.19 *If a finite sequence* $(a_1, a_2, \ldots, a_k)$ *of positive integers satisfies* $a_1 \geq a_2 \cdots \geq a_k$ *and* $a_1 + a_2 + \cdots + a_k = n$*, then we call that sequence a* partition *of the integer* n*.*

Note that, somewhat regrettably, the word *partition* is used both for partitions of the integer n and for partitions of the set $[n]$, which are, of course, very different notions. Some other languages solved this problem by using different words for these two notions. When reading in English, the reader should be careful not to confuse these two concepts. We try to help that effort by stressing that n is an integer, but $[n]$ is a set.

The number of partitions of the integer n is denoted by $p(n)$.

Example 2.20 *The positive integer* 4 *has five partitions, namely,* (4)*,* $(3, 1)$*,* $(2, 2)$*,* $(2, 1, 1)$*, and* $(1, 1, 1, 1)$*. Therefore,* $p(4) = 5$*.*

See Figure 2.4 for the first few values of $p(n)$.

n	$p(n)$
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22

Figure 2.4
The values of $p(n)$ for $n \leq 8$.

At this point, the reader probably cannot wait for us to prove an elegant formula for $p(n)$. While an exact formula for $p(n)$ actually exists, it is by no means simple, to say the least. It involves an infinite sum, complex numbers, the function sinh, and a lot more. The formula was obtained, in various forms, by Hardy, Ramanujan, and Rademacher. The interested reader should consult Chapter 5 of the book by George Andrews, *Theory of Partitions* [6], or the book by George Andrews and Kimmo Eriksson, *Integer Partitions* [7], for details.

While a detailed discussion of this formula for $p(n)$ is way beyond the scope of this book, let us mention at least one reason for which this enumeration problem is more difficult than the previous ones. In the previously discussed problems, the enumeration formulae that we proved were either *exponential* (or even larger) in the relevant variable n, such as 2^{n-1}, or $n!$, or k^n, or *polynomial*, such as $(n)_k$, or $\binom{n}{k}$. However, we will explain in Exercises 19 and 20 that $p(n)$ grows faster than any polynomial function $q(n)$. It would be slightly more cumbersome, but still not too difficult, to prove that, on the other hand, $p(n)$ grows slower than any exponential function a^n, for $a > 1$. (One rough upper bound for $p(n)$ is provided by the number of all compositions of n.) Therefore, $p(n)$ is neither exponential nor polynomial, implying that it must be a more "exotic," less well-known function.

The following theorem, which we will not prove, tells us more about the growth rate of $p(n)$. Recall the notation $\sim$, which was introduced immediately following formula (1.12).

Theorem 2.21 *As $n \to \infty$, the function $p(n)$ satisfies*

$$p(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{2.1}$$

At this point the reader might say that this formula does contain an exponential expression, whereas we promised that $p(n)$ will be smaller than all exponential functions of n. Note that the right-hand side of the above formula is an exponential function of $\sqrt{n}$, and not n. No matter how small a constant a is, the function e^{an} will always grow faster than the right-hand side of the above formula and, therefore, faster than $p(n)$.

Chapter 5 of [6] explains how to deduce this asymptotic formula from the mentioned complicated exact formula for $p(n)$.

At this point, the reader could say that maybe it was too ambitious to start with the quest for a formula of $p(n)$, the number of all partitions of n, without first attacking the problem of counting the partitions of n into a *given* number of parts. This is essentially equivalent to the problem of counting partitions of n into *at most k* parts. Let the number of such partitions be $p_k(n)$. Then Exercises 19 and 20 show that, for any fixed k, the function $p_k(n)$ is a function that is very close to a polynomial. The reader is invited to solve those exercises in order to make that statement more precise. In the rest of this section, we will concentrate on a fascinating proof technique applicable to integer partitions, that of *Ferrers shapes*.