simple, and we will show the proof in Exercise 23. Let us now continue with the proof of our theorem.

If n is pentagonal, then the above arguments showing the bijectivity of g will still work, with two exceptions. That is, if $n = a(3a \pm 1)/2$, and a is even, then, as we saw in cases 1b. and 2, there is one particular partition in $s \in S$ that is not matched with a partition in T, so $|S| = |T| + 1$. On the other hand, if $n = a(3a \pm 1)/2$, and a is odd, then there is one particular partition in $t \in T$ that is not matched with a partition in S, so $|T| = |S| + 1$. We have mentioned that, for any given n, only one of the exceptional situations can occur. Therefore, the proof is complete. $\diamondsuit$

We will return to the pentagonal number theorem in Exercise 16 of Chapter 3, where we will show how it can be used to deduce a recurrence relation for the number of all partitions of n.

Quick Check

1. Prove that the number of partitions of n in which the integer 1 is not a part is equal to $p(n) - p(n-1)$.
2. We select eight partitions of the set [5]. Is it true that there will be two among them so that those two set partitions agree in their number of blocks of size m, for each positive integer m?
3. We select twelve compositions of 6 at random. Is it true that there will be two among them so that if we rearrange the parts of each of them in nonincreasing order, then we get two identical integer partitions?

2.4 The inclusion–exclusion principle

In this section, we are going to investigate situations in which direct methods would lead to an *overcount*, that is, certain objects would be counted more than once, leading to an incorrect final answer.

2.4.1 Two intersecting sets

Let us revisit our friends on the canoe trip whose travails we discussed in Chapter 1. Undaunted by the problems last summer, they are going on a trip again. We hear that this excursion also turns out to be eventful. To be more precise, we hear that five of them fell in the water at one point or another, while nine of them saw their breakfast stolen by raccoons. None of the friends on the trip managed to escape without either of these two experiences. How many friends went on the canoe trip this year?

Before the reader jumps to the conclusion that we want to bore her with the addition principle again, let us point out that this question is *different* from those in Section 1.1. The difference lies in the fact that this time we do *not* know that the two sets of unfortunate friends are disjoint. Indeed, unless we are told otherwise, we cannot assume that falling in the water will protect somebody's breakfast from raccoons.

Continuing this line of thinking, we easily see that we cannot even solve this problem without getting more information. Indeed, let A be the set of people who fell in the water, and let B be the set of those whose breakfast was impounded by raccoons. It could happen that $A \subset B$, which would imply that nine friends went on the trip this year, or it could happen that A and B are disjoint, which, by the addition principle, would imply that 14 friends did, or it could be that the number of people who had both type of problems was not zero, but less than five, which would result in a final answer larger than nine, but less than 14.

As the last paragraph suggests, the missing piece of information is indeed the number of people who belong to both A and B, that is, the number of those friends who belong to $A \cap B$. Let this number be k. Then we could answer the question as follows: There were five people who fell into the water, and there were $9 - k$ whose breakfast was stolen, but who did not fall in the water, therefore, by the addition principle, the number of all people on the trip is $5 + (9 - k) = 14 - k$.

Generalizing the ideas in the above argument, we get the following useful lemma.

Lemma 2.32 *Let A and B be finite sets. Then we have*

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{2.2}$$

Proof: We give a slightly different proof from the one we sketched above, since we will want to generalize our proof later. The left-hand side of (2.2) counts the elements of $A \cup B$. The first two members of the right-hand side do the same, except that elements that belong to both A and B are counted *twice*, once in $|A|$ and once in $|B|$. However, the third term corrects this anomaly, by subtracting precisely the number of elements that belong to both A and B. Therefore, the right-hand side also counts the elements of $A \cup B$, and each of them once, so the two sides of (2.2) count the same objects and are therefore equal. $\diamondsuit$

Example 2.33 *The number of positive integers less than or equal to* 300 *that are divisible by at least one of* 2 *and* 3 *is* 200.

Solution: Let A be the set of those eligible integers that are divisible by 2, and let B be the set of those eligible integers that are divisible by 3. Then $|A| = 150$ and $|B| = 100$. To compute $|A \cap B|$, note that n is divisible by both 2 and 3 if and only if n is divisible by 6. This shows that $|A \cap B| = 50$. So, by Lemma 2.32,

$$|A \cup B| = 150 + 100 - 50 = 200.$$

$\Diamond$

Two intersecting sets, and their sizes, can be visualized with the help of a Venn diagram, shown in Figure 2.13.

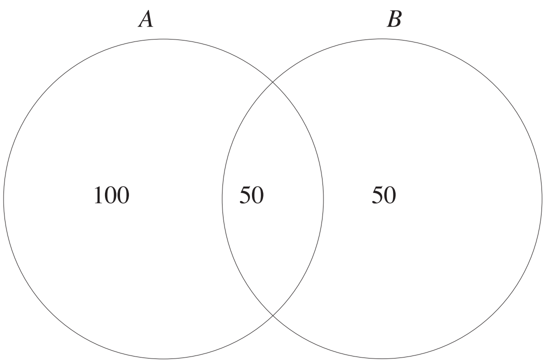

Figure 2.13
The two intersecting sets of Example 2.33.

The previous example showed how to use Lemma 2.32 to compute the number of elements that belong to *at least one* of two sets. Combining that with the subtraction principle, we can find the number of ways to compute the number of elements that do not belong to either one of two sets. We suggest that the reader solve Exercise 13 before reading further.

Example 2.34 *We have invited 30 guests to a wedding. We would like to split them into three groups of sizes of 10, in preparation of seating assignments. However, there are certain guests who do not like each other, namely, U does not like V, and X does not like Y. Therefore, these people cannot be put in the same group. In how many ways can we proceed?*

Solution: Let us count the *bad* partitions first, that is, those partitions of [30] in which each block is of size 10, but either 1 and 2 are in the same block, or 3 and 4 are in the same block, or both of those unfortunate events occur.

Let A be the set of eligible partitions of [30] in which the first bad thing happens, that is, 1 and 2 are in the same block. Removing 1 and 2 from such a partition, we get a partition of [28] into two blocks of size 10, and one block