A combinatorial argument is preferred.

2. Let $A_1, A_2, \cdots, A_n$ be finite sets. Prove that

$$\big| \cup_{i=1}^{n} A_i \big| \geq \sum_{1 \leq i \leq j \leq n} \big| A_i \cap A_j \big|.$$

A combinatorial argument is preferred.

3. What is the number of partitions of $[n]$ into three blocks in which each block is of size at least 2?

2.5 The twelvefold way

In this section, we will review how the counting techniques learned in the first two chapters imply the solutions of 12 counting problems that are easy to formulate in terms that are similar to each other. In each of these problems, we are placing n balls into k boxes. What makes these problems all different is whether the balls are all the same or all distinguishable (for instance, labeled by different numbers), whether the boxes are all the same or all distinguishable, and the conditions for the placement of the balls, which will sometimes require that at most (or at least) one ball be placed in each box.

Gian-Carlo Rota named this collection of problems the *twelvefold way*. In what follows, we will go through these 12 problems one by one. At this point, the reader has all the necessary tools to solve these problems, so the reader should feel free to try to come up with a solution before reading our argument.

In the first three problems, we will assume that our n balls are all *identical*, and that our n boxes are all *identical*.

Problem 2.50

(a) How many ways are there to distribute n identical balls in k identical boxes?

(b) How many ways are there to distribute n identical balls in k identical boxes if each box has to get at least one ball?

(c) How many ways are there to distribute n identical balls in k identical boxes if each box has to get at most one ball?

Solution:

(a) As both the boxes and the balls are identical, all that matters is how many balls get into the individual boxes. In other words, we are dealing with *partitions of the integer n* here. Since the boxes are identical, we might as well order them in nonincreasing order of the balls contained in them. Therefore, the number of all possibilities is $p_k(n)$, since there will be at most k boxes actually containing balls.

(b) Because distributions with less than k nonempty boxes are not allowed, we get $p_k(n) - p_{k-1}(n)$ possibilities. This number is 0 if $n < k$.

(c) This number is 0 if $n > k$ and 1 if $n \leq k$, since each box gets 0 or 1 ball(s). As the boxes are identical, it does not matter which boxes get 0 and which boxes get 1.

◊

In the next three problems, the boxes will still be *identical*, but the balls will be *distinguishable*.

Problem 2.51

(a) How many ways are there to distribute n distinguishable balls in k identical boxes?

(b) How many ways are there to distribute n distinguishable balls in k identical boxes if each box has to get at least one ball?

(c) How many ways are there to distribute n distinguishable balls in k identical boxes if each box has to get at most one ball?

Solution:

(a) Since the balls are all different, we can label them by the elements of $[n]$. Then the balls are put in identical boxes, so we are dealing with *partitions of the set* $[n]$ here. If there are no other restrictions, then we are free to use any number of boxes not exceeding k, so the answer is $\sum_{i=1}^{k} S(n,i)$ by the addition principle.

(b) In this case, we have to use exactly k boxes, so the answer is $S(n,k)$.

(c) If $n > k$, then the answer is 0. Otherwise, the answer is 1, since each nonempty box will have one ball, and it does not matter which box has which ball as the boxes are all the same. Note the similarity with part (c) of the preceding problem.

◊

In the next three problems, we reverse the conditions: The boxes will be *distinguishable*, but the balls will be *identical*.

Problem 2.52

(a) How many ways are there to distribute n identical balls in k distinguishable boxes?

(b) How many ways are there to distribute n identical balls in k distinguishable boxes if each box has to get at least one ball?

(c) How many ways are there to distribute n identical balls in k distinguishable boxes if each box has to get at most one ball?

Solution:

(a) Since the balls are identical, all that matters is how many of them get into each box. However, the boxes are distinguishable, so their order matters. In other words, we are dealing with *weak compositions* of n into k parts. Therefore, the answer is $\binom{n+k-1}{k-1}$.

(b) As no box can be empty, we are dealing with *compositions* of n into k parts, so the answer is $\binom{n-1}{k-1}$.

(c) If $n > k$, then the answer is 0. Otherwise, we simply need to choose the n boxes that will contain one ball each, which we can do in $\binom{k}{n}$ ways.

◊

In the last three problems, both the boxes and the balls will be *distinguishable.*

Problem 2.53

(a) How many ways are there to distribute n distinguishable balls in k distinguishable boxes?

(b) How many ways are there to distribute n distinguishable balls in k distinguishable boxes if each box has to get at least one ball?

(c) How many ways are there to distribute n distinguishable balls in k distinguishable boxes if each box has to get at most one ball?

Solution:

(a) As any ball can go into any box, the number of possibilities is k^n.

(b) This is similar to partitioning $[n]$ into k blocks, but now the order of the blocks matters, so the answer is $S(n,k)k!$.

(c) Again, the answer is 0 if $n > k$. Otherwise, there are k choices for the first ball, then $k-1$ choices for the second ball, and so on, so the answer is $(k)_n$.

◊

If the reader worked through all 12 problems, then she will have a few useful questions to ask when trying to solve an enumeration problem. Are the objects we are counting distinct? Does their order matter? Is repetition