Finally, we mention that, if F is a Ferrers shape on n boxes, and we bijectively label the boxes of F by the elements of $[n]$ so that the labels increase going down and to the right, we get a *Standard Young Tableau.* (We have seen a very special case of this after Lemma 1.32.) These objects have an extremely rich combinatorial structure and are the subject of several books on their own, such as *Young Tableaux* [34], by William Fulton and *The Symmetric Group* [65] by Bruce Sagan.

2.7 Chapter review

(A) Explicit enumeration formulae

1. Number of n-element multisets over $[k]$ is $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.
2. Number of weak compositions n into k parts is $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.
3. Number of compositions of n into k parts is $\binom{n-1}{k-1}$.

(B) Recurrence relations related to set partitions

1. Stirling number of the second kind, triangular recurrence:
$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$
2. Stirling number of the second kind, vertical recurrence:
$$S(n+1,k) = \sum_{i=0}^{n} \binom{n}{i} S(n-i,k-1).$$
3. Bell numbers:
$$B(n+1) = \sum_{k=0}^{n} B(k)\binom{n}{k}.$$

(C) Integer partitions

1. $$p(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

2.8 Exercises

1. A function $f : S \to T$ is called an *injection* if it maps different elements into different elements, that is, if $x \neq y$ implies $f(x) \neq$

$f(y)$. A function $f : S \to T$ is called a *surjection* if, for all $t \in T$, there is an $s \in S$ so that $f(s) = T$.

(a) If a function f is both an injection and a surjection, then it has a name that we have already defined. What is that name?

(b) How many injections $f : [n] \to [k]$ are there?

(c) How many surjections $f : [n] \to [k]$ are there?

2. Prove that the number of n-element multisets over $[k]$ is $\binom{n+k-1}{n}$ using the following approach. For each n-element multiset A over k, write the entries a_i of A in nondecreasing order so that they satisfy the chain of inequalities

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq k.$$

Now observe that n-element subsets $B = \{b_1, b_2, \cdots, b_n\}$ of $[n+k-1]$ satisfy the chain of inequalities

$$1 \leq b_1 < b_2 < \cdots < b_n \leq n + k - 1.$$

Complete the proof by finding a bijection between the sets of n-tuples that satisfy the two chains of inequalities above.

3. Prove a closed formula (no summation signs) for the number of compositions of n into *any* number of parts by

(a) using the formula for the number of compositions of n into a given number of parts, or

(b) by induction on n.

4. Is there a closed formula for the number of all weak compositions of n into any number of parts?

5. We want to divide 12 children into four playgroups. However, there are two pairs of siblings among the children, and we do not want to put siblings in different groups. How many possibilities do we have?

6. What is the number of compositions of 50 into four odd parts?

7. We want to distribute 20 thousand dollars in bonus payments among six workers. Three of the workers have contracts that stipulate that they get at least two thousand dollars each, while each of the remaining workers must get at least one thousand dollars each. How many possibilities do we have if the amount of any bonus will have to be a multiple of 1000?

8. What is the number of weak compositions of 10 into four parts, all of which are smaller than nine?

9. What is the number of *monotone* functions from $[n]$ into $[n]$? We call a function monotone if $x < y$ implies $f(x) \leq f(y)$.

10. What is the number of compositions of 24 into any number of parts so that each part is divisible by three?

11. (a) Prove that, for all positive integers x,

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k. \tag{2.10}$$

    (b) Prove that the result of part (a) holds for all *real numbers* x, not just for positive integers.

12. (Basic knowledge of linear algebra required.) Interpret the result of the previous exercise in terms of bases of the vector space of all polynomials with real coefficients.

13. I want to partition the set [10] into three blocks, so that two blocks are of size 3, and one block is of size 4. In how many different ways can I do this?

14. Five people participate in a long-jump competition. There are no four-way or five-way ties, but there can be two-way or three-way ties. How many possible rankings are there?

15. Prove Theorem 2.18.

16. Prove that the number of partitions of n into at most k parts is equal to the number of partitions of n into parts that are not larger than k.

17. What is the number of partitions of 14 into four distinct parts?

18. A partition that is equal to its own conjugate is called *self-conjugate*. Which positive integers have no self-conjugate partitions?

19. Recall that $p_k(n)$ is the number of partitions of n into at most k parts.

    (a) Prove that, for any fixed k, the function $p_k(kn)$ is a polynomial function of n. What is the degree of this polynomial?

    (b) Prove that, for any fixed k, and for any fixed $r < k$, the function $p_k(kn + r)$ is a polynomial function of n.

20. Use the results of the previous exercise to show that $p(n)$ grows faster than any polynomial function $q(n)$. That is, show that for any polynomial function $q(n)$, there exists a positive integer N_q so that

$$p(n) > q(n) \text{ for all } n > N_q. \tag{2.11}$$

21. Prove Proposition 2.29.

22. Prove Proposition 2.31.

23. If n is a pentagonal number, and $n = a(3a+1)/2$ or $n = a(3a-1)/2$ for a positive integer a, then we will say that a makes n pentagonal. Prove that, for each pentagonal number n, there is only one positive integer a that makes n pentagonal.

24. The *Durfee square* of an integer partition p is the largest square that fits in the northwest corner of the Ferrers shape of p. See Figure 2.15 for an illustration.

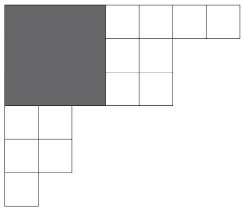

Figure 2.15
The Durfee square of this partition is of size 3×3.

Let the Durfee square of p have side length k. Then we define the *successive ranks* $r_1, r_2, \cdots, r_k$ of p as follows: The successive rank r_i is the difference of the ith row and the ith column of the Ferrers shape of p. So sometimes the successive ranks could be negative. For instance, the successive ranks are 1, 0, 2 for the partition shown in Figure 2.15.

Prove that the number of partitions of n into distinct parts is equal to the number of partitions of $2n$ in which each successive rank is equal to 1.

25. In a company of people, each person writes down the number of other people present that the given person knows. (We assume that if A knows B, then B knows A.)

    (a) Prove that the sum of all the numbers written down is even.

    (b) Let us arrange the numbers written down in nonincreasing order, obtaining a partition p of $2m$ for some integer m. Can the first successive rank r_1 of p be nonnegative?

    (c) Can it happen that $r_1 + r_2 \geq -1$? (Assume that r_2 is defined, that is, the Durfee square of p is of side length at least 2.)

26. Prove that if n is odd, then a partition of n whose third part is 2 cannot be self-conjugate.

27. What is the number of ways to write the digits 1, 1, 2, 2, 3, 4, and 5 in a line so that identical digits are not in consecutive positions?

28. Let us return to Example 2.36. How many ways are there to split the tourists into subgroups if each subgroup is to contain at least one person who speaks the language of the locals?

29. How many positive integers not larger than 1000 are relatively prime to both 7 and 8?

30. The employees of a big company (that means a company with at least three employees) altogether speak four languages. For any three employees, there is at least one language that all of them speak. There is no language that all employees speak. Prove that each employee speaks at least three languages. Construct an example where this happens.

31. Find an alternative argument, not involving computation, for part (b) of the proof of Theorem 2.38.

32. Let $D(n)$ be the number of derangements as defined in Example 2.49. Find a closed formula for the sum

$$\sum_{i=0}^{n} \binom{n}{i} D(i).$$

You can set $D(0) = 1$.

33. Let $A_1, A_2, \cdots, A_n$ be finite sets. Introduce the notation

$$D_k = \sum_{(i_1, i_2, \cdots, i_k)} |A_1 \cap A_2 \cap \cdots \cap A_{i_k}|,$$

where $(i_1, i_2, \cdots, i_k)$ ranges all k-element subsets of $[n]$. In other words, D_k is the sum of the sizes of all k-fold intersections of the sets A_i. Prove that the number of elements that belong to exactly $n-2$ of the A_i is

$$B_{n-2} = D_{n-2} - (n-1)D_{n-1} + \binom{n}{2} D_n.$$

34. Keeping the notations of the previous exercise, prove that the number of elements that belong to exactly $n-3$ of the A_i is

$$B_{n-3} = D_{n-3} - (n-2)D_{n-2} + \binom{n-1}{2} D_{n-1} - \binom{n}{3} D_n.$$

35. State and prove a formula for the number B_k of elements that belong to exactly k sets A_i for some $k \in [n]$.

36. Show an example for an infinite family F of sets so that each infinite subfamily of F has an empty intersection and every finite subfamily of F has an infinite intersection. Try to find at least two solutions.