37. Prove Proposition 2.44. Do not use Theorem 2.48. Use the general version of the inclusion–exclusion principle.

38. Six chess players participate in a tournament. How many final rankings are possible if two-way ties are allowed but ties of more than two players are not?

39. Prove that the average number of blocks taken over all partitions of $[n]$ is
$$\frac{B(n+1)}{B(n)} - 1.$$

40. + Let $f_{n,k}$ be the number of all surjections from $[n]$ to k. Prove that, for any fixed positive integers n, we have
$$\sum_{k=1}^{n}(-1)^k f_{n,k} = (-1)^n.$$

2.9 Solutions to exercises

1. (a) Such a function f is called a "bijection." Indeed, the fact that f is surjective assures that each $t \in T$ has at least one preimage under f, and the fact that f is injective ensures that each $t \in T$ has at most one preimage under f.

   (b) We can consider the elements of $[n]$ as distinguishable balls and the elements of $[k]$ as distinguishable boxes. Then, by part (c) of Problem 2.52, the answer is $(k)_n$.

   (c) Using the same model as in part (b), the answer is $S(n,k)k!$ by part (b) of Problem 2.52.

2. Set $b_i = a_i + i - 1$. The reader is invited to fill in the details.

3. (a) We claim that this number is 2^{n-1}. Indeed, use the result of Exercise 13 of Chapter 1, replacing n by $n-1$.

   (b) For $n = 1$, the statement is true. Now assume it is true for $n-1$, and prove it for n. We are going to construct a 2-to-1 map f from the set $C(n)$ of all compositions of n into the set $C(n-1)$ of all compositions of $n-1$. The map f is defined as follows: Let $c = (c_1, c_2, \cdots, c_k) \in C(n)$. Set
$$f(c) = \begin{cases} (c_1 - 1, c_2, \cdots, c_k) \text{ if } c_1 > 1, \\ \\ (c_2, \cdots, c_k) \text{ if } c_1 = 1. \end{cases}$$

It is easy to see that f is indeed 2-to-1. Indeed, each d element of $C(n-1)$ has two preimages under f; one can be obtained by increasing the first element of d by 1, and the other can be obtained by prepending d by a 1. The reader is asked to verify that the 2^{n-1} compositions of n we obtain this way are indeed all different.

4. There are infinitely many weak compositions of n if the number of parts is not specified. Indeed, we are free to add as many zeros to the end of any weak composition of n as we would like.
5. As the pairs of siblings cannot be separated, let us consider each pair as one "superchild." Then we have 10 children, and we can divide them into four playgroups in any way we want, so we have $S(10,4)$ possibilities.
6. We will consider the equivalent problem of distributing 50 identical balls into four distinguishable boxes so that each box gets an odd number of balls. First we put one ball in each box. Then we have 46 balls left, which we need to distribute into four boxes so that each of the boxes gets an even number of balls. Note that it is not required that each box gets a ball. This is the same as the number of ways to distribute 23 identical balls into four distinguishable boxes, so the number we are looking for is $\binom{23+4-1}{4-1} = \binom{26}{3}$.
7. There is a bijection f from the set S of acceptable distributions into the set T of all weak compositions of 11 into six parts. Indeed, if $a = (a_1, a_2, \cdots, a_6)$ denotes an acceptable distribution, that is, the first three a_i are at least two, and the last three a_i is at least one, then we set

$$f(a) = (a_1 - 2, a_2 - 2, a_3 - 2, a_4 - 1, a_5 - 1, a_6 - 1).$$

   Since the sum of the a_i is 20, the sum of the arguments of $f(a)$ is 11. Therefore, the number of all possibilities is $\binom{11+6-1}{6-1} = \binom{16}{5}$.
8. We apply the subtraction principle here. The number of *all* weak compositions of 10 into four parts is $\binom{13}{3} = 286$. Let us count the *bad* ones among these, that is, those that contain a part 9 or larger. Clearly, there are four weak compositions that contain 10. There are 12 that contain 9. Indeed, 9 can be at four places within the composition, and then the remaining 1 can be at three places, and the product principle implies our claim. Therefore, by the addition principle, there are $4 + 12 = 16$ bad weak compositions of 10, so there are $286 - 16 = 270$ good ones.
9. This is the same as distributing n identical balls (the elements of the domain of our functions) into n distinguishable boxes (the elements of the range of our functions). Once the balls are distributed, we number them 1 through n, but because of the requirement that our function be monotone, we must label them so that the balls in the

first nonempty box get the smallest numbers, the balls in the second nonempty box get the next smallest numbers, and so on. Therefore, the number of monotone functions from $[n]$ into $[n]$ is $\binom{2n-1}{n-1}$.

10. There is a bijection between the set of these compositions of 24 and the set of all compositions of 8. The bijection simply divides each part by three. Therefore, the number we are looking for is simply the number of compositions of 8, which is $2^7 = 128$.

11. (a) We show that both sides count all words of length n that can be built over an x-element alphabet. For the left-hand side, this is a direct consequence of the product principle, or, more precisely, Corollary 1.11. The right-hand side is a little bit more complicated. Let us count those words in which exactly k different letters appear. There are $S(n,k)$ ways to split up the n positions of the word among the k kinds of letters; then there are $(x)_k$ ways to choose the actual letters for these positions. Note that the order of the letters matters; this is why we have not only $\binom{x}{k}$ but $(x)_k$ choices. So, by the product principle, there are $S(n,k)(x)_k$ such words, and then the claim is proved by summing for all k.

    (b) The result of part (a) shows that the two polynomials x^n and $\sum_{k=0}^{n} S(n,k)x^k$ of degree n agree for infinitely many values of x, that is, for all positive integers. That means that their difference is zero infinitely many times. However, their difference is also a polynomial of degree at most n, so the only way it can have more than n roots is when it is the zero polynomial.

12. The set of all polynomials with real coefficients forms a vector space over the field of real numbers. One basis of this field is $\{1, x, x^2, \cdots\}$, and another one is $\{1, (x)_1, (x)_2, \cdots\}$. The result of the previous exercise shows that the *transition matrix* between these two bases is the infinite matrix S formed from the Stirling numbers of the second kind. That is, $S_{n,k} = S(n,k)$ for all nonnegative integers k and n. The first few entries of S are shown below.

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

In other words, if we set $\mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \\ \cdots \end{pmatrix}$ and $\mathbf{x}' = \begin{pmatrix} 1 \\ (x)_1 \\ (x)_2 \\ \cdots \end{pmatrix}$, then we will have $S\mathbf{x}' = \mathbf{x}$.

13. Let us write the elements of [10] in a line, in one of 10! ways, then let us insert a bar after the third and the sixth element. This will certainly give us a partition of the kind we need. However, each partition will be obtained many times. That is, we will get the same partition if we permute the first three elements among each other in 3! ways, the second three elements among each other in 3! ways, and the last four elements among each other in 4! ways. Finally, we can switch the block of the first three elements with the block of the second three elements without changing the partition. Therefore, by the product principle, each partition is obtained in $3! \cdot 3! \cdot 4! \cdot 2$ ways. So, by the division principle, the number of partitions of [10] of the desired kind is
$$\frac{10!}{3! \cdot 3! \cdot 4! \cdot 2}.$$

14. If there are no ties at all, then there are $5! = 120$ possible outcomes.

    If there is one two-way tie and no other ties, then there are $\binom{5}{2} = 10$ choices for the two people in the tie, then there are 4! possible rankings, leading to 240 possibilities by the product principle. If there are two two-way ties, then there are $\binom{5}{2}\binom{3}{2}/2 = 15$ choices for the two pairs in the ties, so there are six possible rankings, leading to 90 different possibilities.

    If there is one three-way tie and no other ties, then we have $\binom{5}{3} = 10$ choices for the triple in the tie, so we have six possible rankings, leading to 60 possibilities. Finally, if there is a three-way tie and a two-way tie, then we have $\binom{5}{3} = 10$ choices for the triple in the tie, so we have two possible rankings, leading to 20 possibilities.

    Therefore, the total number of possible outcomes is
$$120 + 240 + 90 + 60 + 20 = 530.$$

15. Say the block containing the element $n+1$ has $n-k$ additional elements. Then there are $\binom{n}{n-k} = \binom{n}{k}$ ways to choose these elements, so there are $B(k)$ ways to partition the rest of $[n+1]$. The result then follows by the addition principle and summation over all k.

16. We claim that there is a simple bijection between these two sets of partitions, more precisely, between their Ferrers shapes. Indeed, partitions of n into at most k parts correspond to Ferrers shapes on n boxes that have at most k rows. Partitions of n into parts not larger than k correspond to Ferrers shapes on n boxes that have at most k columns. So taking conjugates is a bijection between these two sets.

17. Let (a_1, a_2, a_3, a_4) be such a partition. Then $(a_1-3, a_2-2, a_3-1, a_4)$ is a partition of 8. Vice versa, if (b_1, b_2, b_3, b_4) is a partition of 8,

then $(b_1 + 3, b_2 + 2, b_3 + 1, b_4)$ is a partition of 14 into four distinct parts. Therefore, the number we are looking for is the number of partitions of 8 into four parts. There are five such partitions, namely, $(5, 1, 1, 1)$, $(4, 2, 1, 1)$, $(3, 3, 1, 1)$, $(3, 2, 2, 1)$, and $(2, 2, 2, 2)$.

18. The only such integer is 2. Indeed, if $n = 2k + 1$, then the partition consisting of a row and a column of length $k + 1$ is self-conjugate. If $n = 2k + 4$, then the partition consisting of a row of length $k + 2$, a column of length $k + 2$, and a second row of length 2 is self-conjugate. Both constructions work for $k \geq 0$, leaving $n = 2$ as the only exception. The reader is encouraged to find an alternate proof based on the result of Supplementary Exercise 19.

19. (a) We claim that the degree of this polynomial is $k - 1$, and we are going to prove this, and the fact that $p_k(kn)$ is a polynomial, by induction on k. For $k = 1$, the equality $p_k(kn) = 1$ holds, and the statement is true. Otherwise, we claim that

$$p_k(kn) = p_{k-1}(kn) + p_k((n-1)k). \tag{2.12}$$

Indeed, a partition enumerated by the left-hand side either has fewer than k parts, and then it is enumerated by the first term of the right-hand side, or has exactly k parts, and then subtracting 1 from each of its parts we get a partition enumerated by the second term of the right-hand side. After rearranging, we get

$$p_k(kn) - p_k((n-1)k) = p_{k-1}(kn). \tag{2.13}$$

In other words, the difference of two consecutive values of $p_k(kn)$ is, by the induction hypothesis, a polynomial of degree $k - 2$. This implies that $p_k(kn)$ is a polynomial of degree $k - 1$. The reader is encouraged to prove this proposition by induction on k.

(b) Analogous to part (a), only (2.12) needs to be replaced by

$$p_k(kn + r) = p_{k-1}(kn + r) + p_k((n-1)k + r). \tag{2.14}$$

20. Let us assume that the opposite is true; that is, let us assume that there is a polynomial $q(n)$ for which (2.11) does not hold. Let q have degree $k - 2$. Note that $p(n) > p_k(n)$ for any $n > k$. On the other hand, the previous exercise shows that $p_k(n)$ can be described by k polynomials, each of degree $k - 1$, one for each residue class modulo k. Therefore, even $p_k(n)$ grows faster than $q(n)$, implying that so too does $p(n)$.

Note that a function like $p_k(n)$, that is, a function defined by

$$f(n) = \begin{cases} f_0(n) \text{ if } n = kr, \\ f_1(n) \text{ if } n = kr + 1, \\ \cdots \\ f_{k-1}(n) \text{ if } n = kr + (k-1) \end{cases}$$

where the f_i are polynomials, is called a *quasi-polynomial.* Some people in group theory tend to call them *polynomials on residue classes* or *porcs.*

21. Let $s = (s_1, s_2, \cdots, s_u)$. We added 1 to the first $least(s)$ parts of a. Before that, these parts were consecutive integers; therefore, they will be consecutive after this addition. On the other hand, because of $least(s) < stair(s)$, we originally had $stair(s) = least(s) + 1$, so we had $s_{least(s)+1} = s_{least(s)} - 1$. We did not increase $s_{least(s)+1}$, but we did increase $s_{least(s)}$, so we broke the line of parts that are consecutive integers at the $least(s)$th part. This shows that $k(g(s)) = least(s)$.

    To prove the second statement, note that the parts of s are all distinct. In particular, the next-to-last part s_{u-1} of s is larger than $least(s) = k(g(s))$. The statement then follows as either $l(g(s)) = s_{u-1}$ or $l(g(s)) = s_{u-1} + 1$.

22. The first statement is trivial because we defined the last row of $g(s)$ by taking one box from each of the first $stair(s)$ rows. To see the second one, note that decreasing each element of a decreasing sequence of consecutive integers by 1 will result in another decreasing sequence of consecutive integers.

23. First, it is impossible to have $a(3a + 1) = b(3b + 1)$ or $a(3a - 1) = b(3b - 1)$ because both functions $f(m) = m(3m + 1)$ and $g(m) = m(3m - 1)$ are strictly monotone on the positive integers. Second, we cannot have $a(3a + 1) = b(3b - 1)$ either, since that equation is equivalent to $b - a = \frac{1}{3}$.

24. The number of partitions of n into distinct parts is the same as the number of partitions of $2n$ into distinct even parts (just multiply each part by 2). Let S be the set of the latter, and let T be the set of partitions of $2n$ in which each successive rank is 1. We define a bijection $f : S \to T$ as follows: Let $s \in S$, and let s_1 be the first part of s_1. Then s_1 is even, say $s_1 = 2h$. Start building a Ferrers shape by taking a first row of length $h + 1$ and a first column of length h. This will take precisely $2h = s_1$ boxes because of the overlap in the corner. Then continue on this way. That is, if the second part of s is $s_2 = 2i$, then add a second row of length $i + 1$ and a second column of length i (not counting the boxes already in the shape) to

the shape we build, and so on. When we use up all rows of s, we will get a Ferrers shape on $2n$ boxes in which each successive rank is 1. Let $f(s)$ be the partition that belongs to that Ferrers shape. See Figure 2.16 for an example. We leave the easy task of proving that f is a bijection to the reader.

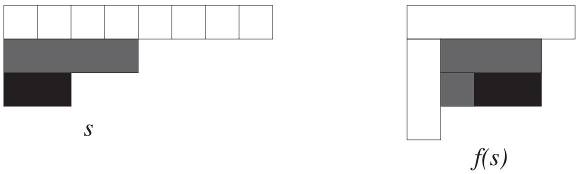

Figure 2.16
Parts become stripes.

25. (a) Let us say that during the party every pair of people who know each other shakes hands once. If the total number of handshakes is m, then the sum we are interested in is $2m$ since each handshake takes two people. In other words, when we sum the numbers written down, each handshake gets counted twice, once for each person who participated in that handshake.
    (b) No. Let A be the person who knows most people in the company, and let the company have n people. Then the length of the first row of p is simply the number of other people A knows (so less than n), while the length of the first column is simply n. Therefore, $r_1 \leq -1$.
    (c) No. Let B be the person who knows most people in the company except for A. Then the length of the second row is the number $|B|$ of people that B knows. Let $|A| + |B| = n + t$. That means there are at least t people who know both A and B. Therefore, the length of the second column is at least $t + 2$ (since A and B know at least two people as well). Therefore, the sum of the first two rows is $n + t$, while the sum of the first two columns is at least $n + t + 2$, so $r_1 + r_2 \leq -2$.

    Note that even more is true. The famous Erdős–Gallai criterion says that the partitions that can play the role of p are precisely those in which

$$r_1 + r_2 + \cdots + r_i \leq -i,$$

    for all $i \in [k]$, where k is the side length of the Durfee square of p.

26. Because the third part of p is 2, the Durfee square of p is of size 2. Since taking conjugates leaves the 2×2 Durfee square fixed, it follows that the number of boxes in a self-conjugate Ferrers shape with this Durfee square is even.

27. We can instead count the permutations of these digits in which two identical digits get into consecutive positions. Let us first count those, the elements of the set A, in which the two 1s are in consecutive positions. Replacing the two 1s by a symbol S, we see that $|A| = 6!/2$. Similarly, if B is the set of permutations of our digits in which the two 2s are in consecutive positions, then $|B| = 6!/2$ by the same argument. Finally, $|A \cap B| = 5!$, since we can replace the two 1s by an S, and the two 2s by a T. So Lemma 2.32 shows that $|A \cup B| = 6! - 5!$. Therefore, by the subtraction principle, the number of permutations of our multiset in which identical digits are not in consecutive positions is
$$\frac{7!}{2 \cdot 2} - (6! - 5!).$$

28. Clearly, all partitions of [10] that were bad for the purposes of Example 2.36 are bad here, too. However, there are additional bad partitions, namely, those in which $\{1,2\}$ is a block, or $\{1,3\}$ is a block, or $\{2,3\}$ is a block, or $\{1,2,3\}$ is a block. Fortunately, these sets of partitions are pairwise disjoint, so, by the addition principle, their total number is $3B(8) + B(7)$. Therefore, the total number of bad partitions is this number plus the number of bad partitions from Example 2.36, that is, $3B(9) + 2B(7)$. So the number of good partitions is $B(10) - 3B(9) - 2B(7)$.

29. Let A_1 be the set of integers in [1000] that are not relatively prime to 7; let A_2 be the set of those that are not relatively prime to 8, or, equivalently, are even. Then $|A_2| = 500$. And since A_1 consists of positive integers divisible by 7, we have $|A_1| = 1000/7 = 142$. Finally, $A_1 \cap A_2$ is the set of positive integers in [1000] that are divisible by 14, so $|A_1 \cap A_2| = 1000/14 = 71$. Therefore, by Lemma 2.32, we get
$$|A_1 \cup A_2| = 142 + 500 - 71 = 571.$$

30. We claim that each employee has to speak at least three languages. Let us assume that the converse is true, that is, Albert speaks only English and Spanish. There is someone at the company who does not speak English, say Bella. Then Bella must speak Spanish, or Albert, Bella, and any third person would violate the conditions. However, there is someone at the company who does not speak Spanish, say Christine, and then Albert, Bella, and Christine have no language in common.

    For an example for such a company, let the company have any number of employees, and divide them into five nonempty groups, A, B, C, and D. Let us say that A is the group of people not speaking English, B is the group of people not speaking Spanish,

C is the group of people not speaking German, D is the group of people not speaking French, and E is the group of people speaking all four. Because everyone speaks at least three languages, nobody belongs to two groups. Then, no matter how we choose three people there will be at least one group among the first four from which we did not choose anyone, so all three chosen people will speak the language not spoken in that group.

31. If x does not belong to any A_i, then none of the terms of the right-hand side counts it, so nor does their sum.

32. We have

$$\begin{aligned}\sum_{i=0} \binom{n}{i} D(i) &= \binom{n}{n-i} D(i) \\ &= n!.\end{aligned}$$

Indeed, there are $n!$ permutations of length n. There are $\binom{n}{n-i} D(i)$ of them in which there are exactly $n-i$ fixed points and in which the remaining i entries form a derangement.

33. We are going to show that the right-hand side also counts the elements that belong to exactly $n-2$ of the A_i. If x is contained in exactly $n-2$ of the A_i, then x is counted once by the first term of the right-hand side and zero times by the two other terms. If x is contained by exactly $n-1$ of the A_i, then x is counted $\binom{n-1}{n-2} = n-1$ times by the first term, and then subtracted $n-1$ times by the second term. The third term counts x zero times, so, altogether, x is counted once. Finally, if x belongs to all of the A_i, then the first term counts x exactly $\binom{n}{n-2} = \binom{n}{2}$ times, the second term subtracts x exactly $(n-1)\binom{n}{n-1} = (n-1)n$ times, and the third term counts x exactly $\binom{n}{2}$ times. However, we have

$$\binom{n}{2} - (n-1)n + \binom{n}{2} = 0,$$

completing the proof of our claim that the right-hand side counts each element x with the desired property exactly once.

34. The proof is very similar to that of the previous exercise. One checks the same way that each element x that is contained in exactly $n-3$, or $n-2$, or $n-1$ of the A_i is counted exactly zero times on the right-hand side. For elements that belong to all A_i, this follows from the identity

$$\binom{n}{3} - (n-2)\binom{n}{2} + \binom{n-1}{2} n - \binom{n}{3} = 0.$$

35. We claim that

$$B_k = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} D_i.$$

Let x be an element that is contained in exactly t sets A_i. Then there are three possibilities.

(a) If $t < k$, then none of the D_i on the right-hand side counts x, so x is counted by the right-hand side zero times.

(b) If $t = k$, then the only term of the right-hand side that counts x is the one indexed by $i = k$, and that term is equal to $\binom{k}{k} D_k = D_k$, so x is counted exactly once by the right-hand side.

(c) If $t > k$, then, for any i satisfying $k \leq i \leq t$ there are $\binom{i}{k}$ different i-fold intersections of the A_i counted by D_i. Therefore, x is counted

$$\sum_{i=k}^{t} \binom{t}{i} \binom{i}{k} (-1)^{i-k}$$

times by the right-hand side. Using Supplementary Exercise 16b of Chapter 1, we see that the above expression is equal to

$$\begin{aligned} \sum_{i=k}^{t} \binom{t}{k} \binom{t-k}{i-k} (-1)^{i-k} &= \binom{t}{k} \left(\sum_{i=k}^{t} \binom{t-k}{i-k} (-1)^{i-k} \right) \\ &= \binom{t}{k} (1-1)^{t-k} = 0, \end{aligned}$$

where in the last step we used the binomial theorem and the fact that $t - k$ is positive.

36. Let F_i be the set of all positive integers divisible by i. Then no matter how we select an infinite number of F_i, their intersection is always empty since no positive integer has infinitely many divisors. On the other hand, if we only select a finite number of F_i, say $i_1, i_2, \cdots, i_n$, their intersection is infinite since it contains all positive integers divisible by $i_1 i_2 \cdots i_n$.

Alternatively, let $F_i = [i, \infty)$. It is straightforward to verify that both requirements are satisfied.

37. Let A_i be the set of elements of $[n]$ that are divisible by p_i. Then $|A_i| = n/p_i$, and, in general,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}| = \prod_{\substack{s \\ s \notin \{i_1, i_2, \cdots, i_r\}}} p_s.$$

Applying the principle of inclusion–exclusion, the result now follows, since, by standard algebraic computations,

$$\prod_{i=1}^{t} (p_i - 1) = \sum_{j=0}^{t} (-1)^j \sum_{K} \prod_{k \in K} p_k,$$