3

Generating functions

In this chapter, we will learn our most advanced counting technique so far. We will learn how to encode *all elements* of a possibly *infinite* sequence by *one* single function, the generating function of the sequence. Often, we will first obtain the generating function of a sequence and then decode it, that is, we will then compute the elements of the sequence from the generating function. This idea is a powerful example of using continuous objects in discrete mathematics.

Let us start with a very short review of the type of functions we will use. Rest assured that the book will return to discussing combinatorics very soon.

3.1 Power series

The reader is undoubtedly a good student, and, therefore, the reader surely remembers *power series* from calculus. These are much like polynomials, that is, sums of various powers of the variable x multiplied by constants. But, unlike polynomials, power series can be infinitely long. So, for instance, $\sum_{k=0}^{n} x^k$ is a polynomial, but $\sum_{k=0}^{\infty} x^k$ is a power series.

3.1.1 Generalized binomial coefficients

One frequently encountered example of power series is the binomial theorem with real exponents. In order to review that theorem, we first have to extend the notion of the binomial coefficient $\binom{a}{k}$ to all *real* numbers a (the number k still has to be a nonnegative integer). After all, we cannot define $\binom{-2.5}{k}$ to be the number of all k-element subsets of the set $[-2.5]$, because the set $[-2.5]$ does not exist.

While the combinatorial definition of $\binom{a}{k}$ does not survive if a is not a nonnegative integer, the analytic definition will.

Definition 3.1 *Let a be any real number, and let k be a nonnegative integer. Then we set*

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}.$$

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For example,

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}}{3!} = \frac{3}{48} = \frac{1}{16}.$$

We are now ready to recall the binomial theorem for real exponents from calculus.

Theorem 3.2 *Let a be a real number, and let $|x| < 1$. Then*

$$(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \cdots . \tag{3.1}$$

Proof: Recall that the *Taylor series expansion* of an analytic function $f(x)$ about $x = 0$ is given by

$$f(x) = \sum_{n\geq 0} f^{(n)}(0) \frac{x^n}{n!}.$$

Let us apply this fact with $f(x) = (1+x)^a$. Computing the first few values of $f^n(0)$, we see that $f^{(0)}(0) = f(0) = 1^a = 1$, $f^{(1)}(0) = a$, $f^{(2)}(0) = a(a-1)$, $f^{(3)}(0) = a(a-1)(a-2)$, and so on. From this, one can see the trend $f^{(n)} = (a)_n$, which is then straightforward to prove by induction, using only the rules of differentiation. Since $(a)_n/n! = \binom{a}{n}$, this proves the theorem. ♢

If a happens to be a nonnegative integer, then $\binom{a}{k} = 0$ for $k > a$, so the right-hand side is a finite sum of terms of the type $c_i x^i$; in other words, it is a polynomial. If a is not a positive integer, however, then the right-hand side is an *infinite sum* of the form $\sum_{i\geq 0} c_i x^i$ and is called a *power series.*

In calculus, you must have spent many fun-filled hours trying to find out for which values of x certain power series converged. Some power series converged for all values of x, such as $\sum_{n\geq 0} x^n/n!$. Some others only converged for one value of x, and that value was zero in many examples, such as for $\sum_{n\geq 0} n! x^n$. You learned various tests, such as the ratio test and the root test, that help to determine the values of x for which a given power series is convergent.

When a power series converges for all $x \in [a, b]$, then there is a chance that we can write that power series in a closed form.

For example, the reader may remember that the power series $F(x) = \sum_{n\geq 0} x^n$ converges for $x \in (-1, 1)$ and, in that interval,

$$F(x) = \frac{1}{1-x}. \tag{3.2}$$

The reader is invited to verify that this is in fact a special case of the binomial theorem, with $-x$ replacing x, and $a = -1$.

Power series will be our main tool in this chapter. We will like them so much that we will not even mind if they are not convergent at any interval; we will still work with them.