different ways, that

$$\sum_{n\geq 0}(n+1)x^n = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}. \tag{3.4}$$

Indeed, on one hand, we know from (3.2) that $\sum_{n\geq 0} x^n = 1/(1-x)$. Differentiating both sides, we get the statement to be proved.

On the other hand, the right-hand side is the product of $A(x) = 1/(1-x) = \sum_{n\geq 0} x^n$ and $B(x) = 1/(1-x) = \sum_{n\geq 0} x^n$. That is, we have

$$\frac{1}{(1-x)^2} = \left(\sum_{n\geq 0} x^n\right) \cdot \left(\sum_{n\geq 0} x^n\right).$$

Using formula (3.3), with $a_n = b_n = 1$ for all n, we see that the coefficient of x^n in $A(x)B(x)$ is $\sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} 1 = n+1$, again proving our statement.

If the reader is wondering whether there is a third way to prove (3.4), then our answer is in the affirmative; a third proof can be deduced from the binomial theorem (with real exponents).

Quick Check

1. Find the formal power series form of $1/(1+x)$.
2. Find the formal power series form of $\ln(1+x)$.
3. Find the formal power series form of $\ln(1/(1-x))$.

3.2 Warming up: Solving recurrence relations

We start our study of the applications of generating functions by using them to solve recurrence relations. While many recurrence relations can be solved without generating functions, this will be a good warm-up for the more sophisticated applications of our techniques.

3.2.1 Ordinary generating functions

Five people wearing red hats start spreading a rumor that, if you do not put on a red hat, you will become sick. Every hour, each of them tells the rumor to two other people, who immediately all believe it, put on red hats, and enlist to spread the rumor further, with the same speed. No person will be told the rumor twice; this is what the red hats are for. At the same time, one person each hour will discontinue to believe in the rumor, will take his red hat off, and will stop spreading the rumor. Otherwise, nobody else will stop spreading the rumor. How many people will wear red hats after n hours?

When hearing this question, it is natural to try and see what happens within the first few hours. Let a_n denote the number of people wearing red hats at the end of hour n. Then we know that $a_0 = 5$, and that

$$a_n = 3a_{n-1} - 1 \tag{3.5}$$

if $n \geq 1$. This is because each person brings in two followers *in addition to himself*, and because one person takes his red hat off.

Formula (3.5) is enough to compute the values of $a_1, a_2, \cdots$. Indeed, $a_1 = 3a_0 - 1 = 14$, and $a_2 = 3a_1 - 1 = 41$. We could certainly continue this way, but that would become boring and tedious rather soon. Furthermore, it would not be very efficient either. Indeed, assume we only want to know what a_{200} is. Then, using the above step-by-step method, we would first have to compute $a_1, a_2, \cdots, a_{199}$, and it would be only *then* that we could compute a_{200}. Our goal is to find a formula that provides the value of a_n without asking for the value of a_{n-1} or other elements of the sequence $\{a_i\}_{i\geq 0}$. (For the rest of this book, we use the notation $\{a_i\}_{i\geq 0}$ for the sequence $a_0, a_1, \cdots$.)

The crucial idea is that we can encode all elements of the sequence $\{a_i\}_{i\geq 0}$ by one single power series. This leads to the technique of generating functions, which is the central topic of this chapter and an extremely useful method of combinatorial enumeration.

Definition 3.3 *Let $f_0, f_1, \cdots$ be a sequence of real numbers. Then the formal power series*

$$F(x) = \sum_{n\geq 0} f_n x^n$$

is called the ordinary generating function *of the sequence $\{f_i\}_{i\geq 0}$.*

The following example just repeats a fact we already know and have recently mentioned, this time in the language of generating functions.

Example 3.4 *Let $f_n = 1$ for all $n \geq 0$. The ordinary generating function of this sequence is*

$$F(x) = \sum_{n\geq 0} f_n x^n = \sum_{n=0} x^n = \frac{1}{1-x}.$$

We urge the reader to take the time and justify the summation above. (Hint: Multiply both sides of $\sum_{n=0} x^n = \frac{1}{1-x}$ by $1 - x$ and see what cancels on the left-hand side.)

Example 3.5 *Let $f_n = n$. Then*

$$\begin{aligned} F(x) &= \sum_{n=0} f_n x^n = x \sum_{n\geq 1} n x^{n-1} = x \left(\frac{1}{1-x} \right)' \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

In the above two examples, we were given a sequence by an explicit formula, and we then computed the generating function using that formula. Often, as in our running example with red hats and rumors, we have to do the converse, that is, find an explicit formula for a sequence if its generating function is known. Before we attack the problem of spreading rumors, let us see an example for this reverse computation.

Example 3.6 *Find an explicit formula for f_n if*

$$F(x) = \sum_{n\geq 0} f_n x^n = \frac{x}{1-3x}.$$

Solution: The crucial observation is that f_n is nothing other than the coefficient of x^n in $F(x)$. So we have to find a formula for that coefficient or, equivalently, for the coefficient of x^{n-1} in $\frac{1}{1-3x}$. However, replacing x with $3x$ in Example 3.4, we have

$$\frac{1}{1-3x} = \sum_{n\geq 0} (3x)^n = \sum_{n\geq 0} 3^n \cdot x^n.$$

Therefore, the coefficient we are looking for is $f_n = 3^{n-1}$. $\Diamond$

Our techniques in this chapter will very often ask for the coefficient of x^n in a certain formal power series $A(x)$. Therefore, we introduce the notation $[x^n]A(x)$ for that coefficient. So the result of the previous example can be expressed with this notation by writing $[x^n]F(x) = 3^{n-1}$.

In our running example, we are looking for an explicit formula for the numbers a_n. To that end, we first need to compute their ordinary generating function $A(x) = \sum_{n\geq 0} a_n x^n$. In order to be able to do that, we need an identity that is satisfied by $A(x)$. We can get such an identity as follows: Take the defining recurrence relation of our sequence, that is, the equation $a_n = 3a_{n-1} - 1$, and multiply both sides by x^n. Then sum over all values of n for which the defining equation holds; in our case, that means for $n \geq 1$. We then get

$$\sum_{n\geq 1} a_n x^n = 3 \sum_{n\geq 1} a_{n-1} x^n - \sum_{n\geq 1} x^n.$$

Now note that the left-hand side is almost exactly $A(x)$; just the term a_0 is missing. That is, the left-hand side is $A(x) - a_0 = A(x) - 5$. Similarly, note that the first term of the right-hand side is nothing but $3xA(x)$, and that the second term of the right-hand side is just $x/(1-x)$. (Consider Example 3.4 again.) So the last identity implies

$$A(x) - 5 = 3xA(x) - \frac{x}{1-x},$$

$$A(x)(1-3x) = 5 - \frac{x}{1-x},$$

or, solving for $A(x)$,

$$A(x) = \frac{5}{1-3x} - \frac{x}{(1-x)(1-3x)}. \tag{3.6}$$

Just as in Example 3.6, we can now find an explicit formula for a_n by finding an explicit formula for $[x^n]A(x)$. According to (3.6), this coefficient is equal to the coefficient of x^n in $\frac{5}{1-3x}$ minus the coefficient of x^n in $\frac{x}{(1-x)(1-3x)}$, so we need to compute these two coefficients. The former is very easy to compute since

$$\frac{5}{1-3x} = 5 \cdot \frac{1}{1-3x} = 5 \cdot \sum_{n \geq 0} 3^n x^n,$$

so the coefficient of x^n in this power series is $5 \cdot 3^n$.

It is a little bit more difficult to compute the coefficient of x^n in the second term of the right-hand side of (3.6), that is, in $\frac{x}{(1-x)(1-3x)}$.

There are several ways to get around this problem. Perhaps the fastest is to decompose $\frac{x}{(1-x)(1-3x)}$ into the sum of partial fractions. That is, we can look for real numbers P and Q so that

$$\frac{x}{(1-x)(1-3x)} = \frac{P}{1-x} + \frac{Q}{1-3x}.$$

After multiplying both sides by $(1-x)(1-3x)$, we get

$$x = P + Q - x(3P+Q).$$

So we must have $P + Q = 0$, and $3P + Q = -1$, leading to $P = -1/2$ and $Q = 1/2$. Therefore, we have

$$\begin{aligned} \frac{x}{(1-x)(1-3x)} &= \frac{1}{2} \cdot \frac{1}{1-3x} - \frac{1}{2} \cdot \frac{1}{1-x} \\ &= \frac{1}{2} \sum_{n \geq 0} 3^n x^n - \frac{1}{2} \sum_{n \geq 0} x^n = \sum_{n \geq 0} \frac{3^n - 1}{2} x^n. \end{aligned}$$

The coefficient of x^n in this power series is $(3^n - 1)/2$, so that is the coefficient of x^n in $\frac{x}{(1-x)(1-3x)}$.

Therefore,

$$[x^n]A(x) = a_n = 5 \cdot 3^n - \frac{3^n - 1}{2}. \tag{3.7}$$

So this is the number of people wearing red hats after n hours.

Note that we could have used the fact that

$$[x^n]\frac{x}{(1-x)(1-3x)} = [x^{n-1}]\frac{1}{(1-x)(1-3x)},$$

and then we could have decomposed the latter into a sum of partial fractions.

We have therefore succeeded in finding an explicit formula for a_n. The reader is invited to verify that this formula is indeed correct, first by computing the first few values of a_n by this formula and finding that we obtain 5, 14, 41, as we should, and then by a more general argument that shows that our formula is indeed correct for all n. See Exercise 2.

Why did the idea of generating functions help? The defining equation (3.5) had the drawback of containing two unknowns, a_n and a_{n-1}. The generating function $A(x)$, however, comprised *all* values of a_n, so when we translated (3.5) into an equation containing $A(x)$ (namely, (3.6)), then the only unknown in that equation was $A(x)$. Therefore, we could get an explicit expression for $A(x)$ from that equation. Once we knew $A(x)$, the determination of a_n was a matter of computation.

Let us practice this method by considering another example.

Example 3.7 *Five people wearing red hats start spreading a rumor. In the first hour, each of them tells the rumor to one person who did not wear a red hat before. These new converts will put on red hats and enlist to spread the rumor further. Then the same trend continues, following the rule that each person will tell the rumor to one other person in his first hour after enlisting, and to nine other people in each subsequent hour. Nobody will ever take their red hat off. How many people will wear red hats after n hours?*

Let b_n be the number of people wearing a red hat after n hours. Then $b_0 = 5$, and $b_1 = 10$. Just as before, we start by finding a recursive rule satisfied by the numbers b_n. What do we know about b_n? First, in the nth hour, every person counted by b_{n-1} tells the story to one other person, resulting in $2b_{n-1}$ people wearing red hats. Second, in the nth hour, those people who are also counted by b_{n-2}, that is, those who are not fresh converts, tell the story to eight more people each (besides those already accounted for), resulting in $8b_{n-2}$ additional people wearing red hats. This leads to the identity

$$b_n = 2b_{n-1} + 8b_{n-2}, \tag{3.8}$$

for all $n \geq 2$.

The next step is to define the ordinary generating function $B(x) = \sum_{n\geq 0} b_n x^n$ and to translate (3.8) into an equation containing $B(x)$. To that end, let us multiply both sides of (3.8) by x^n and sum over all n for which

(3.8) holds, that is, for $n \geq 2$. We get

$$\begin{aligned}\sum_{n\geq 2} b_n x^n &= 2\sum_{n\geq 2} b_{n-1}x^n + 8\sum_{n\geq 2} b_{n-2}x^n \\ &= 2x\sum_{n\geq 2} b_{n-1}x^{n-1} + 8x^2\sum_{n\geq 2} b_{n-2}x^{n-2}.\end{aligned}$$

Now is the time to look for expressions close to $B(x)$ on both sides of the last equation. We find that the last equation is equivalent to

$$B(x) - 15x - 5 = 2x(B(x) - 5) + 8x^2B(x),$$

$$B(x)(1 - 2x - 8x^2) = 5,$$

$$B(x) = \frac{5}{1 - 2x - 8x^2}.$$

In order to find the power series form of $B(x)$, we use the technique of partial fractions as learned in calculus and sketched in our previous example. If you feel confident that you remember that technique, you can skip the following paragraph.

We are in the fortunate situation that the denominator factors nicely, namely, $1 - 2x - 8x^2 = (1 + 2x)(1 - 4x)$. Therefore, we have

$$\frac{5}{1 - 2x - 8x^2} = \frac{C}{1 + 2x} + \frac{D}{1 - 4x},$$

$$5 = C(1 - 4x) + D(1 + 2x),$$

$$5 = x(2D - 4C) + C + D.$$

The only way that the (constant) polynomial 5 can be equal to the polynomial $x(2D - 4C) + C + D$ is by the coefficient of x in the latter being 0 and the constant term in the latter being 5. This leads to $D = 10/3$ and $C = 5/3$. Therefore, we get

$$B(x) = \frac{5}{1 - 2x - 8x^2} = \frac{5}{3}\cdot\frac{1}{1 + 2x} + \frac{10}{3}\cdot\frac{1}{1 - 4x}. \tag{3.9}$$

Using (3.4), this leads to

$$\begin{aligned}B(x) &= \frac{5}{3}\sum_{n\geq 0}(-2x)^n + \frac{10}{3}\sum_{n\geq 0}(4x)^n \\ &= \frac{5}{3}\sum_{n\geq 0}(-1)^n\cdot 2^n x^n + \frac{10}{3}\sum_{n\geq 0}4^n x^n \\ &= \sum_{n\geq 0}\left(\frac{5}{3}(-1)^n\cdot 2^n + \frac{10}{3}4^n\right)x^n.\end{aligned}$$

Finally, we obtain $b_n = [x^n]B(x)$, that is,

$$b_n = \frac{5}{3}(-1)^n \cdot 2^n + \frac{10}{3}4^n. \tag{3.10}$$

One can check again that this formula is indeed correct for all $n \geq 0$. The reader is asked to do so in Supplementary Exercise 2.

The last two recurrence relations, that is, (3.5) and (3.8), were reasonably easy to solve because, when we multiplied them by x^n and added over all values of n for which these equations held, we got expressions in which it was quite easy to recognize the generating function of the relevant sequence. It was either just multiplied by a constant or multiplied by a power of x. Sometimes, we have to do a little more to find our generating functions in the last step. The following example illustrates this.

Example 3.8 *Let $a_0 = 1$, and let us assume that*

$$\binom{n+2}{2} = \sum_{i=0}^{n} a_i a_{n-i} \tag{3.11}$$

for all integers $n \geq 1$. Find an explicit formula for a_n.

This recurrence relation is somewhat unusual in that a_n depends on all previous values of a_i, not just a few. We can compute directly that $a_1 = 3/2$ and $a_2 = 15/8$.

Solution: Note that (3.11) holds even for $n = 0$. Now multiply both sides of (3.11) by x^n, and add over all $n \geq 0$, that is, all values of n for which the equality holds. We get

$$\sum_{n\geq 0} \binom{n+2}{2} x^n = \sum_{n\geq 0} \sum_{i=0}^{n} a_i a_{n-i} x^n.$$

This is the moment to look for the generating function $A(x) = \sum_{n\geq 0} a_n x^n$ on both sides. However, we have to be a little more resourceful than before. Observe that the left-hand side is precisely $(1-x)^{-3}$. (Details are given in Exercise 1.) As far as the right-hand side is concerned, observe that this is precisely $A(x)^2$. Indeed, apply formula (3.3) with $B(x) = A(x)$. Therefore, the last displayed equation simply says that

$$(1-x)^{-3} = A(x)^2,$$
$$(1-x)^{-3/2} = A(x).$$

In order to compute the coefficients of $A(x)$, we will need to compute the binomial coefficient $\binom{-3/2}{n}$. We get that

$$\begin{aligned}\binom{-3/2}{n} &= \frac{\frac{-3}{2} \cdot \frac{-5}{2} \cdot \cdots \cdot \frac{-2n-1}{2}}{n!} \\ &= (-1)^n \frac{1 \cdot 3 \cdots (2n+1)}{2^n \cdot n!}.\end{aligned}$$