Quick Check

1. Let $a_0 = 1$ and let $a_n = 5a_{n-1} - 1$ if $n \geq 1$. Find an explicit formula for a_n.
2. Let $b_1 = 0$, and let $b_{n+1} = (n+1)b_n + n$ if $n \geq 1$. Find an explicit formula for b_n.
3. Let $a_0 = 0$, let $a_1 = 1$, and let $a_n = 4a_{n-1} - 3a_{n-2}$ if $n \geq 2$. Find an explicit formula for a_n.

3.3 Products of generating functions

In this section, we treat problems where generating functions are even more essential than they were in solving recurrence relations in the previous section.

3.3.1 Ordinary generating functions

Example 3.12 *A utility crew is painting n poles next to a road, starting at one end of the road and advancing one by one. For each pole, they randomly choose a color out of their three possibilities (red, blue, and green). At a certain time, they notice that their shift will end soon, and they will not be able to paint more poles that day. In order to celebrate the end of the workday, they choose one of the remaining poles at random and paint a smiling face on it, in red. How many different looks can the poles of this road have after the happy crew has left?*

The difference between the situation of this example and the situations studied in the previous sections is that the workers here proceed according to *two* different rules, one before realizing time was up, and one after that. Furthermore, we do not know when this change occurs, that is, after how many painted poles the crew will choose an unpainted pole for the smiling face. Let us assume that this happens after i poles were painted. That means the first i poles are painted in one of 3^i ways; then the crew chooses a remaining pole, in one of $n-i$ ways. So by the product principle the street can have $3^i(n-i)$ different looks. See Figure 3.2.

Let a_n denote the number of ways the poles can be after the crew left. Using the addition principle for $i \in [0, n]$, the argument of the previous paragraph shows that

$$a_n = \sum_{i=0}^{n} 3^i(n-i). \tag{3.16}$$

With the natural initial condition $a_0 = 0$ (if there are no poles, there is no pole for the smiling face, and the crew will not come anyway), this leads to the first values $a_1 = 1$ and $a_2 = 5$.

i painted poles 3^i	one chosen pole $n - i$

Figure 3.2
The product principle at work.

We would now like to find an explicit formula for a_n. To that end, note that the right-hand side of (3.16) looks very much like the coefficient of x^n in a *product of two generating functions*. In order to make this observation more precise, set $A(x) = \sum_{n\geq 0} a_n x^n$, and also, $B(x) = \sum_{n\geq 0} 3^n x^n = \frac{1}{1-3x}$, and $D(x) = \sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$. As we mentioned, the crucial observation is that

$$A(x) = B(x)D(x). \tag{3.17}$$

Indeed, the coefficient of x^n on the right-hand side is $\sum_{i=0}^{n} 3^i(n-i)$, which is in turn equal to a_n by (3.16).

Therefore, we can obtain a_n as $[x^n]A(x) = [x^n]B(x)D(x)$. Fortunately, we know both $B(x)$ and $D(x)$ explicitly. So (3.17) yields

$$\begin{aligned} A(x) &= B(x)D(x) = \frac{1}{1-3x} \cdot \frac{x}{(1-x)^2} \\ &= \frac{x}{(1-3x)(1-x)^2}. \end{aligned}$$

In order to find the power series form of $A(x)$, we decompose the last expression as a sum of partial fractions as we learned in calculus. That is, we are looking for real numbers P, Q, and R so that

$$A(x) = \frac{x}{(1-3x)(1-x)^2} = \frac{P}{1-3x} + \frac{Q}{1-x} + \frac{R}{(1-x)^2}.$$

A routine computation yields that $P = 3/4$, $Q = -1/4$, and $R = -1/2$. Thus we have

$$\begin{aligned} A(x) &= \frac{3}{4}\frac{1}{1-3x} - \frac{1}{4}\frac{1}{1-x} - \frac{1}{2}\frac{1}{(1-x)^2} \\ &= \frac{3}{4}\sum_{n\geq 0} 3^n x^n - \frac{1}{4}\sum_{n\geq 0} x^n - \frac{1}{2}\sum_{n\geq 0}(n+1)x^n \\ &= \sum_{n\geq 0} \frac{3^{n+1} - 1 - 2(n+1)}{4} x^n. \end{aligned}$$

Therefore, we get that $a_n = \frac{3^{n+1}-1-2(n+1)}{4}$. The reader is invited to verify that this formula is indeed correct.

The most important feature of the previous example was that the crew *started doing something*, then, at one point, it *did something else*. This idea of cutting the interval $[1, n]$ into two parts and then considering distinct structures on each part occurs in many problems. Fortunately, the *product* of the ordinary generating functions counting the two structures is a very powerful tool in these situations.

Let us consider a similar problem where the same line of thinking helps.

Example 3.13 *A student makes a study plan for the final exam period. She splits the period into two parts. In the first part, she devotes each day either to physics or to abstract algebra, except for one day, when she will only study the chosen subject of the day in the morning, and relax in the afternoon. She devotes all days of the second part to combinatorics, except for two days, when she takes breaks. These two days do not have to be consecutive days. If the exam period consists of n days, in how many different ways can the student plan her studies?*

Solution: Let us assume again that the first part of studies will consist of k days and the second part will consist of $n-k$ days. That means that the student has $k2^k$ possibilities for the first part and $\binom{n-k}{2}$ possibilities for the second part. So, by the product principle, the student has $k2^k\binom{n-k}{2}$ possibilities for this fixed k. Summing over all allowed k (which means $1 \leq k \leq n-2$, otherwise there are not enough days for breaks), we see that the total number of possibilities she has is $\sum_{k=1}^{n-2} k2^k\binom{n-k}{2}$. Following the line of thinking of the previous example, consider the generating functions of the numbers of possibilities in each part. That is, consider the generating functions

$$B(x) = \sum_{n\geq 1} n2^n x^n = \frac{2x}{(1-2x)^2},$$

and

$$D(x) = \sum_{n\geq 2} \binom{n}{2} x^n = \frac{x^2}{(1-x)^3}.$$

Let $A(x) = B(x)D(x)$. Similarly to the previous example, the number of all possibilities for the student, that is, the number $\sum_{k=1}^{n-2} k2^k\binom{n-k}{2}$, is precisely the coefficient of $[x^n]A(x)$. So all we have to do is to find that coefficient. Using the previous two equations, we get

$$\begin{aligned} A(x) &= B(x)D(x) = \frac{2x}{(1-2x)^2} \cdot \frac{x^2}{(1-x)^3} \\ &= \frac{2x^3}{(1-2x)^2(1-x)^3}. \end{aligned}$$

Converting this generating function into partial fraction form is a tedious task.

However, there are various software packages that can do it for us. Either way, we get that

$$\begin{aligned} A(x) &= \frac{2}{(1-x)^3} + \frac{2}{(1-x)^2} + \frac{6}{1-x} + \frac{2}{(1-2x)^2} - \frac{12}{1-2x} \\ &= \sum_{n\geq 0}(n+2)(n+1)x^n + \sum_{n\geq 0} 2(n+1)x^n + \sum_{n\geq 0} 6x^n \\ &+ \sum_{n\geq 0}(n+1)2^{n+1}x^n - \sum_{n\geq 0} 12\cdot 2^n x^n \\ &= \sum_{n\geq 0}(2^{n+1}(n-5) + (n+1)(n+4) + 6)x^n. \end{aligned}$$

Therefore, the number of possibilities the student has is $a_n = 2^{n+1}(n-5) + (n+1)(n+4) + 6$. The reader is invited to verify that this is indeed the correct formula, in particular, $a_1 = a_2 = 0$ and $a_3 = 2$. ♢

Our result illustrates the power of the method very well. After all, the formula $a_n = 2^{n+1}(n-5) + (n+1)(n+4) + 6$ would have been quite difficult to guess, and it would have been quite difficult to prove the formula combinatorially. There seems to be no easy explanation for the summands $2^{n+1}(n-5)$, or $(n+1)(n+4)$, or 6 in the formula. However, our generating function method worked fairly mechanically, and we did not need any bright ideas.

We have seen two examples of the same counting principle, and now we are going to formulate the general theorem applicable to similar situations.

Theorem 3.14 (Product formula) *Let f_n be the number of ways one can carry out a certain task on the set $[n]$. Let g_n be the number of ways one can carry out another task on $[n]$. Let $F(x)$ and $G(x)$ be the ordinary generating functions of the sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$.*

Let h_n be the number of ways to split the set $[n]$ into the intervals $\{1, 2, \cdots, i\}$ and $\{i+1, i+2, \cdots, n\}$, and then carry out the first task on the first interval and the second task on the second interval. Let $H(x)$ be the ordinary generating function of the sequence $\{h_n\}_{n\geq 0}$. Then

$$H(x) = F(x)G(x). \tag{3.18}$$

Proof: The definition of the numbers h_n yields the recurrence relation

$$h_n = \sum_{i=0}^{n} f_i g_{n-i}.$$

Indeed, for any fixed i there are f_i ways to carry out the first task, g_{n-i} ways to carry out the second task, and so there are $f_i g_{n-i}$ ways to carry out both of them. Summing over all i, the claim follows by the addition principle.

Now note that h_n is the coefficient of x^n in $H(x)$, and that $\sum_{i=0}^{n} f_i g_{n-i}$ is the coefficient of x^n in $F(x)G(x)$. As these coefficients are equal for all n, the power series $H(x)$ and $F(x)G(x)$ are also equal. ♢

Note that Example 3.12 was a straightforward application of the product formula. Indeed, the first task, to be carried out on the first i poles, was to color each of them red, blue, or green, leading to the generating function $F(x) = \sum_{n\geq 0} 3^n x^n = \frac{1}{1-3x}$. The second task, to be carried out on the last $n-i$ poles, was to choose one pole, leading to the generating function $G(x) = \sum_{n\geq 0} nx^n = \frac{x}{(1-x)^2}$. As we now know without needing further explanation, the generating function of the combined task was then simply $F(x)G(x) = \frac{x}{(1-x)^2(1-3x)}$.

Let us consider one more simple application of the product formula before attacking more difficult problems.

Example 3.15 *A section in a combinatorics textbook contains n exercises. The book is used by a dutiful professor who wants the students to practice the basic methods discussed in the section, but also wants to provide a challenge for those who are interested in more difficult questions. Therefore, she will assign the first $2i$ problems for homework (for some integer $i \geq 2$) and also one of the remaining exercises as a bonus problem. In how many different ways can she proceed?*

Solution: In the language of the product formula, the professor splits the set $[n]$ into two intervals, then she assigns all problems in the first interval as homework (which she can do in one way if the size of that interval is even and at least four, and in zero ways otherwise), and chooses one problem from the second interval (which she can do in j ways if that is the size of that interval).

Therefore, again with the notations of the product formula,

$$F(x) = x^4 + x^6 + \cdots = \sum_{i\geq 2} x^{2i} = x^4 \sum_{i\geq 0} (x^2)^i = \frac{x^4}{1-x^2}$$

and

$$G(x) = \sum_{j\geq 0} jx^j = \frac{x}{(1-x)^2}.$$

By the product formula, this yields

$$H(x) = F(x)G(x) = \frac{x^5}{(1-x^2)(1-x)^2}.$$

While there is still work to be done, we can see that we are on the right track since $H(x)$ will not have terms with exponents less than five; indeed, the professor could not make her choices if the section had less than five exercises.

In order to find $[x^n]H(x)$, it suffices to find $[x^{n-5}]\frac{1}{(1-x^2)(1-x)^2}$. This is a routine, if tedious computation, which can also be done by computer. (We will talk about that a little bit more in the next example.) Either way, we get

$$\frac{H(x)}{x^5} = \frac{1}{2(1-x)^3} + \frac{1}{4(1-x)^2} + \frac{1}{8(1-x)} + \frac{1}{8(1+x)}.$$

Using the identity proved in Exercise 1, this yields

$$\begin{aligned} \frac{H(x)}{x^5} &= \frac{1}{2}\sum_{n\geq 0}\binom{n+2}{2}x^n + \frac{1}{4}\sum_{n\geq 0}(n+1)x^n + \\ &+ \frac{1}{8}\left(\sum_{n\geq 0}x^n + \sum_{n\geq 0}(-1)^n x^n\right) \\ &= \sum_{n\geq 0}\frac{4\binom{n+2}{2} + 2(n+1) + 1 + (-1)^n}{8}x^n. \end{aligned}$$

So the number h_n of possibilities that the professor has is the coefficient of x^{n-5} in the latest expression, that is, $h_n = \frac{4\binom{n-3}{2}+2(n-4)+1+(-1)^{n-5}}{8}$. ♢

Let us now practice the use of the product formula on a classic problem that is more difficult. Once we see the numerical solution of the problem, it will remind us of something we have seen before.

Example 3.16 *Two soccer teams played an exciting game last weekend. The game ended in a draw, with each team scoring* n *goals. Throughout the game, it often seemed likely that the home team would win, since they often had the lead, while the visitors never did. In how many different orders could the* $2n$ *goals be scored?*

Solution: So that we can understand the problem better, let us compute the answer h_n to the above question for small values of n. Denote by A a goal scored by the home team, and denote by B a goal scored by the visiting team. If $n = 1$, then the only possible order of goals is AB, since the home team never trailed. If $n = 2$, then there are two possibilities, namely, $AABB$ and $ABAB$. If $n = 3$, then we have five possibilities, namely, $AAABBB$, $AABABB$, $AABBAB$, $ABAABB$, and $ABABAB$. So $h_1 = 1$, $h_2 = 2$, and $h_3 = 5$. Also note that $h_0 = 1$ since there is one way for both teams not to score (by not scoring). Let $H(x) = \sum_{n\geq 0} h_n x^n$.

What makes this problem more difficult than the previous problems or some of the standard counting problems of Chapter 1 is the condition that the home team *never trailed*. Without that condition, there would be $\binom{2n}{n}$ possibilities, since we would only need to count words of length $2n$ consisting of n copies of A and n copies of B. Now we also have the condition that no initial segment of our words can contain more Bs than As.

We would like to use the product formula to compute $H(x)$. The difficulty lies in finding out how to decompose $H(x)$ into the product of two generating functions. What are the two tasks mentioned in the product formula?

Let us assume the game did not end in a scoreless tie (0–0). Let us call the moment when the visitors first tied the score the *critical moment* of the game. Since the game did end in a draw, we know for sure that there was a critical moment. Let us say that, at the critical moment, the score was a tie at i goals each.

We will now break the game up into two parts, the part before the critical moment and the part after the critical moment.

First, we claim that there are h_{n-i} ways the game could be completed *after* the critical moment. Indeed, simply cancel all goals scored before the critical moment, and then the remainder of the game has to end in a tie of $n-i$ goals for each team, with the home team never trailing. Therefore, the number of ways the game could be completed after the critical moment is enumerated by the generating function

$$G(x) = H(x).$$

It is a little bit more interesting to figure out how many ways the goals could be scored *before* the critical moment. Note that it would be incorrect to say that the number of ways this could happen is h_i since the visiting team never tied before the critical moment, and h_i does not take that into account.

Instead, observe the following: The home team must have scored the first goal, and the visiting team must have scored the last goal before the critical moment. Let us call the time period *between* these two goals, and excluding them, the *middle period* of the game. So the first goal of the game came before the middle period, and the critical moment came after the middle period. (Soccer games do not really have middle periods; they have two halves, but never mind.) Then the middle period resulted in a tie at $i-1$ goals for each team, so that the home team never trailed *when counting goals scored in this period*. Indeed, to say that the home team never trailed when counting goals scored in the middle period is the same as saying that considering the whole game the visitors did not tie the game before the critical moment. This is because the home team has the lead when the middle period starts. Therefore, the number of ways to reach the critical moment with an $i-i$ tie is the same as the number of middle periods resulting in an $(i-1)-(i-1)$ tie, that is, h_{i-1}.

So there are h_{i-1} ways to go from the start of the game to the critical moment if the critical moment comes at the score of $i-i$. Therefore, the generating function for the number of possibilities for the part of the game before the critical moment is

$$F(x) = \sum_{i\geq 1} h_{i-1}x^i = xH(x).$$

We are now ready to use the product formula. Do not forget that our decomposition only works if $n > 0$ (if there are no goals at all, there is no critical moment). The generating function for the sequence $h_1, h_2, \cdots$ is $H(x)-1$, since $h_0 = 1$. Therefore, the product formula yields

$$H(x) - 1 = xH(x)^2. \tag{3.19}$$

This is a quadratic equation for $H(x)$. Solving it, we get

$$H(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{3.20}$$

We hope that the alert reader is upset that we did not explain how $\pm\sqrt{1-4x}$ in the solution of the quadratic equation (3.19) became simply $-\sqrt{1-4x}$. Here is our explanation: The numbers h_n are, of course, uniquely determined by n; therefore, the generating function $H(x)$ is also unique. So only one of the power series $\frac{1-\sqrt{1-4x}}{2x}$ and $\frac{1+\sqrt{1-4x}}{2x}$ can be equal to $H(x)$. In order to see which one, we need to look no further than at $H(x)$. From the definition of $H(x)$, we see that $H(0) = h_0 = 1$. On the other hand, the function $\frac{1+\sqrt{1-4x}}{2x}$ is not even defined in $x = 0$ since its denominator is 0 while its numerator is 2. The function $\frac{1-\sqrt{1-4x}}{2x}$ can, however, be extended to $x = 0$, since at that point both the numerator and the denominator of the fraction are equal to 0. One easily checks by l'Hôpital's rule or otherwise that $\lim_{x=0} \frac{1-\sqrt{1-4x}}{2x} = 1$, so the choice of the negative sign is justified. (In fact, once we know that the positive sign is not the right one, we know that the negative sign is the right one since $H(x)$ does exist, and it does have to be one of $\frac{1-\sqrt{1-4x}}{2x}$ and $\frac{1+\sqrt{1-4x}}{2x}$.)

In order to find the numbers h_n, we need to find $[x^n]H(x)$. To that end, we first compute $\sqrt{1-4x}$ by the binomial theorem. We get

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n. \tag{3.21}$$

Now we need to compute $\binom{1/2}{n}$. By the definition of binomial coefficients, we get

$$\binom{1/2}{n} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-2n+3}{2}}{n!} = (-1)^{n-1} \frac{(2n-3)!!}{2^n \cdot n!}.$$

Comparing this with (3.21) yields

$$\sqrt{1-4x} = -\sum_{n\geq 0} \frac{2^n \cdot (2n-3)!!}{n!} x^n. \tag{3.22}$$

$$= 1 - 2\sum_{n\geq 0} \frac{\binom{2n-2}{n-1}}{n} x^n. \tag{3.23}$$

Finally, we can substitute the obtained expression into (3.20) to get the power series form of $H(x)$. This yields

$$H(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} \frac{\binom{2n}{n}}{n+1} x^n. \qquad (3.24)$$

Therefore, $H_n = \frac{\binom{2n}{n}}{n+1}$. Note that this shows that the numbers h_n are in fact equal to the Catalan numbers c_n, which we defined after Example 1.34. $\diamondsuit$

Remarks:

1. Note that (3.19) is equivalent to the recurrence relation

$$h_n = \sum_{i=0}^{n-1} h_i h_{n-1-i} \qquad (3.25)$$

for $n \geq 1$ and $h_0 = 1$. In fact, an alternative way to prove the result of Example 3.16 is to prove (3.25) first, then multiply both sides by x^n, sum for $n \geq 1$, then recognize that the left-hand side is $H(x) - 1$ and the right-hand side is $xH(x) \cdot H(x)$.

2. We have mentioned that the scoring sequences counted in this example are equinumerous to northeastern lattice paths from $(0, 0)$ to (n, n) that never go above the diagonal $x = y$. In fact, there is a simple bijection between these two sets, with letters A corresponding to an east step, and letters B corresponding to a north step. Note that the *critical moment* corresponds to the first time a northeastern lattice path touches the diagonal $x = y$.

In order to extend the range of the product formula, note that there is nothing magical about the number *two* when we split $[n]$ into two intervals. If we split $[n]$ into three consecutive intervals so that they are disjoint and their union is $[n]$, and then we carry out various tasks on each interval, we can argue similarly. That is, if $A(x)$, $B(x)$, and $C(x)$ are the generating functions enumerating the number of ways we can carry out the three tasks, then, by the product formula, $A(x)B(x)$ is the generating function for the number of ways to carry out the *first two* tasks. Then, applying the product formula again, for $A(x)B(x)$ and $C(x)$ we get that $A(x)B(x)C(x)$ is the generating function for carrying out all three tasks. We can proceed similarly if we have more than three tasks. This leads to a new array of applications, which are introduced by the following example.

Example 3.17 *Find the number h_n of ways one can pay n dollars using only 1-dollar, 2-dollar, and 5-dollar bills. The order in which we use the bills does not matter, only the number of bills of each denomination does.*

Solution: Let a_n (resp. b_n, c_n) be the number of ways to pay n dollars using only 1-dollar bills (resp. only 2-dollar bills, only 5-dollar bills). Let $A(x)$ (resp. $B(x)$, $C(x)$) be the ordinary generating functions of the corresponding sequences. Then $a_n = 1$ for all $n \geq 0$, while $b_n = 1$ if n is divisible by two and $b_n = 0$ otherwise, and $c_n = 1$ if n is divisible by five, and $c_n = 0$ otherwise. This leads to the generating functions $A(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}$, $B(x) = \sum_{n\geq 0} x^{2n} = \sum_{n\geq 0}(x^2)^n = \frac{1}{1-x^2}$, and $C(x) = \sum_{n\geq 0} x^{5n} = \sum_{n\geq 0}(x^5)^n = \frac{1}{1-x^5}$.

Therefore, by the product formula, the generating function $H(x) = \sum_{n\geq 0} h_n x^n$ of the numbers h_n is

$$H(x) = A(x)B(x)C(x) = \frac{1}{(1-x)(1-x^2)(1-x^5)}. \tag{3.26}$$

◊

Several comments are in order. First, you could ask what is the use of this result since we did not get a formula for the numbers h_n, and getting one by finding the partial fraction decomposition of $\frac{1}{(1-x)(1-x^2)(1-x^5)}$ by hand would be quite some work. As we mentioned before, standard mathematics software, such as Mathematica or Maple, can do this tedious work for us very easily. For instance, in Maple we can type

```
convert(1/((1-x)*(1-x^2)*(1-x^{5})), parfrac);
```

and the computer will return

$$\frac{1}{4(x-1)^2} - \frac{13}{40(x-1)} + \frac{x^3+2x^2+x+1}{5(x^4+x^3+x^2+x+1)} + \frac{1}{8(x+1)} - \frac{1}{10(x-1)^3}.$$

Now it is relatively painless to find the power series form of h_n, especially if we notice that we can multiply both the numerator and the denominator of the middle term by $x-1$, turning the denominator into $5(x^5-1)$.

If we are simply interested in the first 15 values of the sequence, we can get that by simply typing

```
series(1/((1-x)*(1-x^2)*(1-x^{5})), x=0, 16);
```

in Maple and hitting Return. Maple will then return

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 7x^8 + 8x^9 + 10x^{10}$$

$$+11x^{11} + 13x^{12} + 14x^{13} + 16x^{14} + 18x^{15} + O(x^{16}),$$

and we find $h(n)$ as the coefficient of x^n in the above expression for $n \leq 15$.

Second, and this is the more far-reaching of our comments, note that we only found the generating function for the numbers of *partitions* of the non-negative integer n into parts equal to one, two, or five. It goes without saying that it is easy to generalize our result for the cases when other parts are allowed. Because of the broad applications of this method, it is worth spending some time exploring the connection between $H(x)$ and partitions of n into parts of size one, two, and five.

We have seen that

$$H(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots).$$

After carrying out all multiplications, a typical summand on the right-hand side will be of the form $x^i x^{2j} x^{5k} = x^n$. That means that $i + 2j + 5k = n$, or, in other words, that n has a partition into $k + j + i$ parts, k of which are 5, j of which are 2, and i of which are 1. Each partition of n into parts of allowed sizes will correspond to one summand equal to x^n, showing again that the coefficient of x^n in $H(x)$ is the number of partitions of n into parts of size one, two, and five.

You should verify your understanding of the above example by trying to find the summands on the right-hand side that correspond to partitions of 5 into allowed parts.

There are plenty of fascinating theorems on partitions of integers that can be proved by generating functions. The reader is invited to solve Supplementary Exercises 9, 10, 11, and 12 to warm up. Then the reader is invited to solve Exercises 12, 13, and 14 to obtain a very interesting identity involving partitions. Exercises 27, 28, and 29 generalize that identity.

Some of these exercises will involve *infinite* products of infinite sums. This may be unusual for the reader. Therefore, we will discuss the concept here. We will do this using an example, as opposed to in full generality, in order to avoid excessive notation.

Let $p_{even}(n)$ be the number of partitions of n into even parts only, and let $p_{even}(0) = 1$. We claim that then the equality

$$\sum_{n\geq 0} p_{even}(n)x^n = (1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots \quad (3.27)$$

$$= \prod_{i\geq 1} \frac{1}{1 - x^{2i}} \quad (3.28)$$

holds.

First of all, how do we know that the right-hand side, which is an infinite product of infinite sums, is well-defined? Let us turn that question around. What could possibly go wrong? When would the right-hand side not make sense? The right-hand side is a product of sums. We would start computing this sum as if it were a finite sum, that is, by computing the (finite) term-by-term products and then adding them. As we compute finite term-by-term products, such as $x^2 \cdot x^{10} \cdot x^{14}$, each of these products is defined. The problem could come afterward, when we add these products. If infinitely many of these products were equal to x^k for some k, then we could not add them. (We will *not* identify infinite sums with their limit here, even if that limit exists.)