Therefore, the number of ways to make all necessary selections is $h_n = n(3^{n-1} - (-1)^{n-1})/4$, for all $n \geq 1$. ♢

Quick Check

1. How many words of length n are there over the alphabet $\{A, B, C\}$ that consist of an initial part of letters A only, but contain no letters A after that? The initial part does not have to be proper, that is, it is allowed to be of length 0 or n.
2. How many words of length n are there over the alphabet $\{A, B, C\}$ that consist of an initial part and ending part of letters A only, and a middle part that does not contain the letter A? We allow each of the three parts to be empty.
3. How many ways are there to select an odd-sized subset of $[n]$, and then color each of the unselected elements of $[n]$ red, blue, or green?

3.4 Compositions of generating functions

In this section, we extend the reach of our generating function techniques even further.

3.4.1 Ordinary generating functions

We start by exploring a situation when we still subdivide the set $[n]$ into intervals, but in a more complex way.

Example 3.23 *The dissertation of a doctoral student in combinatorics has to consist of exactly n pages. The dissertation can have* as many chapters as the student likes, *but each chapter must contain at least one page of text and at least one page of illustrations. Furthermore, the number of text pages as well as the number of illustration pages in any chapter must be an integer.*

Find an explicit formula for the number b_n of ways in which the student can arrange the pages of his dissertation.

What is difficult in this problem is that we do not know the number of chapters the dissertation will have. If we knew that there would be two chapters, or three chapters, we could easily apply the product formula (for ordinary generating functions) to get the answer.

Indeed, let a_n be the number of ways *one* chapter can be arranged. We claim that $a_n = 2^n - 2$ for $n \geq 2$, and $a_n = 0$ for $n < 2$. This is because each page can be a text page or an illustration page, but the student is not allowed to have text pages only or illustration pages only. Define $A(x) = \sum_{n\geq 0} a_n x^n$, the ordinary generating function of the sequence $\{a_n\}$. Then

$$A(x) = \sum_{n\geq 1}(2^n - 2)x^n = \sum_{n\geq 1} 2^n x^n - 2\sum_{n\geq 1} x^n \tag{3.30}$$

$$= \frac{2x}{1-2x} - \frac{2x}{1-x} = \frac{2x^2}{(1-2x)(1-x)}. \tag{3.31}$$

If the thesis were to consist of two chapters, then the product formula would imply that the possible arrangements the student has are enumerated by the generating function $A^2(x)$.

Therefore, if the thesis were to contain *one or two* chapters, then the possibilities would be enumerated by the generating function $A(x) + A^2(x)$. We can certainly continue this way; that is, if the thesis is to consist of k chapters, then the number of possibilities is given by the formal power series $A^k(x)$, and if there are at most k chapters, then it is given by the power series $\sum_{n=1}^{k} A^n(x)$. If we agree that there is one way to write a thesis with no chapters, then we can say that $A^0(x) = 1$, and then $\sum_{n=0}^{k} A^n(x)$ is the generating function for the number of possibilities of a thesis of at most k chapters.

This is all very good, you might say, but where does this lead us? After all, we do not know how many chapters the thesis will have, and the above method seems to depend on that information. Fortunately, that is not really true. In order to understand why, note that the infinite sum

$$\sum_{n=0}^{\infty} A^n(x) = 1 + A(x) + A^2(x) + A^3(x) + \cdots \tag{3.32}$$

is actually a *well-defined* power series. That is, even if we add infinitely many power series, in their sum the coefficient of x^m is *finite* for all m. (This discussion of infinite sums of power series will probably remind the reader of the discussion of infinite products of power series, which can be found following (3.27).) Indeed, the term of the lowest exponent in $A^n(x)$ is x^{2n}. Therefore, for any m, there is only a finite number of power series of the form $A^n(x)$ that do have an x^m-term, namely, those in which $2n \leq m$. So the coefficient of x^m in our infinite sum $\sum_{n=0}^{\infty} A^n(x)$ is just a sum of *a finite number* of coefficients.

Therefore, the infinite sum $\sum_{n=0}^{\infty} A^n(x)$ does make sense, and, by our previous argument, it is the generating function for the number of ways an n-page long thesis can be arranged with zero chapters, or one chapter, or two chapters, or *any number of chapters*, since the summands do not stop coming. So at the very least, we did find the generating function of the numbers we were looking for. The power series $B(x) = \sum_{n=0}^{\infty} A^n(x)$ is precisely the ordinary generating function $\sum_{n=0}^{\infty} b_n x^n$, where b_n is the number of ways an n-page dissertation can be arranged into any number of chapters, each of which contains at least one text page and at least one page of illustrations.

Again, you might ask, what good is that? What can we do with an infinite sum of infinite sums (power series)? Fortunately, the answer to this question is not nearly as bad as one might expect. The key observation is that

$$B(x) = \sum_{n=0}^{\infty} A^n(x) = \frac{1}{1-A(x)}.$$

However, we do have an exact formula for $A(x)$, since we computed it in (3.30). Using that formula, we get

$$B(x) = \frac{1}{1-A(x)} = \frac{1}{1-\frac{2x^2}{(1-2x)(1-x)}}.$$

In other words, now we do have an explicit formula for the generating function of the numbers b_n! Finding the numbers b_n is now just a matter of computation. Using partial fractions, we find that

$$\begin{aligned} B(x) &= \frac{1}{1-\frac{2x^2}{(1-2x)(1-x)}} = \frac{(1-x)(1-2x)}{1-3x} \\ &= \frac{2}{9}\cdot\frac{1}{1-3x} - \frac{6x-7}{9} \\ &= \frac{2}{9}\cdot\left(\sum_{n\geq 0} 3^n x^n\right) - \frac{2}{3}x + \frac{7}{9} \\ &= 1 + 2\sum_{n\geq 2} 3^{n-2}x^n. \end{aligned}$$

That is, if the dissertation will have n pages, with $n \geq 2$, then there are $b_n = 2\cdot 3^{n-2}$ ways to arrange its pages into chapters.

This is a very nice and compact formula for b_n, and we challenge the reader in Exercise 20 to find a proof for it that does not use generating functions.

The existence of a proof that does not use generating functions for a formula does not decrease the value of a proof for that same formula that does use generating functions. It is often the case, just as in Exercise 20, that a combinatorial proof simply *verifies* the truthfulness of a formula, while a generating function proof *deduces* it. In other words, for a combinatorial proof, we may need to know the formula in advance, while for a generating function proof, we do not.

The novelty of the above computation was that we substituted the *power series* $A(x)$ for x in the power series $1/(1-x)$. That is, we *composed* generating functions. Let us formalize this concept.

Definition 3.24 *Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$ be a formal power series, and let $A(x)$ be a formal power series having constant term 0. Then the* composition *of these power series is the power series*

$$F(A(x)) = \sum_{n=0}^{\infty} f_n A^n(x),$$

where we set $A^0(x) = 1$.

Note that this is a meaningful definition since the coefficient of x^n in $F(A(x))$ is finite for all n. Indeed, the smallest exponent in $A^m(x)$ is at least m, so, if $m > n$, then $A^m(x)$ will not have an x^n-term. So $[x^n]F(A(x))$ is the sum of a finite number of finite summands.

Now that we have formalized the technique of composing formal power series, we may generalize the central idea behind the previous example.

Theorem 3.25 *Let* a_k *be the number of ways to carry out a certain task on a* k*-element set, with* $a_0 = 0$*, and let* $A(x) = \sum_{k\geq 0} a_k x^k$.

Let b_n *be the number of ways to split the interval* $[n]$ *into any number of disjoint nonempty subintervals, then carry out the task enumerated by the numbers* a_i *on each of the subintervals. Set* $b_0 = 1$*, and* $B(x) = \sum_{n\geq 0} b_n x^n$.

Then

$$B(x) = \frac{1}{1 - A(x)}.$$

Proof: By the product formula, the number of ways to split $[n]$ into k nonempty intervals and then carry out the task counted by $A(x)$ on each of them is given by the generating function $A^k(x)$.

There is a subtle point here. The product formula does *not* require that the intervals into which we split $[n]$ be nonempty. However, by requiring this here, we are not cheating since we have $a_0 = 0$ anyway. That is, the formal power series $\sum_{k\geq 0} a_k x^k$ and $\sum_{k\geq 1} a_k x^k$ are identical. On the other hand, for the present problem we must require that the intervals be nonempty, otherwise there would be infinitely many ways to split $[n]$ into disjoint intervals by just adding empty intervals. This is one reason why the requirement that $a_0 = 0$ is essential.

Summing over all k, we get that, if there is no restriction on the number of possibilities, the number of ways to split $[n]$ into disjoint nonempty subintervals and then carry out the original task on each of them is counted by the power series

$$\sum_{k\geq 0} A^k(x) = \frac{1}{1 - A(x)} = B(x).$$

Here, the left-hand side is a well-defined sum as $a_0 = 0$. This shows the other reason for the requirement that $a_0 = 0$. $\Diamond$

Example 3.26 *This fall, the United States Congress will have n working days, split into an unspecified number of sessions. Within each session, one day will be designated for a plenary session, and each of the remaining days (if there are any remaining days) will be designated for either committee work or subcommittee work.*

Find an explicit formula for the number b_n of ways in which Congress can schedule its season.

Solution: This is a situation in which Theorem 3.25 applies, the scheduling of each session being the first task. Let a_n denote the number of ways to schedule one session. Then $a_n = n \cdot 2^{n-1}$, and, therefore,

$$A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} n \cdot 2^{n-1} x^n = \frac{x}{(1-2x)^2}.$$

The last step can be obtained from the identity $\sum_{n\geq 0} nx^{n-1} = 1/(1-x)^2$, which we proved after stating formula (3.4).

Therefore, if b_n is the number of ways to carry out the composite task (splitting the fall into sessions, then specifying the type of work to be done on each working day) and $B(x) = \sum_{n\geq 0} b_n x^n$, then, by Theorem 3.25,

$$\begin{aligned}
B(x) &= \frac{1}{1-A(x)} = \frac{1}{1-\frac{x}{(1-2x)^2}} \\
&= \frac{(1-2x)^2}{(1-2x)^2 - x} = 1 + \frac{x}{(1-2x)^2 - x} \\
&= 1 + \frac{x}{4x^2 - 5x + 1} = 1 + \frac{1}{3} \cdot \frac{1}{1-4x} - \frac{1}{3} \cdot \frac{1}{1-x} \\
&= 1 + \frac{1}{3} \cdot \sum_{n\geq 0} 4^n x^n - \frac{1}{3} \cdot \sum_{n\geq 0} x^n = 1 + \sum_{n\geq 0} \frac{4^n - 1}{3} x^n.
\end{aligned}$$

Consequently, the number of ways to schedule the fall legislative session is $b_n = (4^n - 1)/3$ for $n \geq 1$. $\Diamond$

Now that the reader knows the formula for b_n, we challenge the reader to find an inductive proof for it without the use of generating functions in Supplementary Exercise 19.

Example 3.27 *Let f_n be the number of compositions of n into parts that are 1 or 2. Set $f_0 = 1$. Find the closed form of the generating function $F(x) = \sum_{n\geq 0} f_n x^n$.*

Solution: We cut $[n]$ into nonempty intervals, and then we "identify" each obtained interval as an interval of length one or length two. We can do that in one way if the given interval is *actually* of length one or two, and in zero ways otherwise. That shows that $A(x) = x + x^2$, and therefore, by Theorem 3.25,

$$F(x) = \frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots .$$

$\diamondsuit$

The numbers f_n are called the *Fibonacci numbers.* The reader is asked to prove that they satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Theorem 3.25 can be generalized in the following way.

Theorem 3.28 *(Compositional formula for ordinary generating functions) Let a_n be the number of ways to carry out a task on an n-element set, and set $a_0 = 0$. Let b_n be the number of ways to carry out another task on an n-element set, with $b_0 = 1$. Finally, let c_n be the number of ways to split $[n]$ into nonempty intervals, then carry out the first task on each interval, and then carry out the second task* on the set of these intervals. *Then, if $A(x)$, $B(x)$, and $C(x)$ denote the ordinary generating functions of the three sequences, then the equality*

$$C(x) = B(A(x)) \tag{3.33}$$

holds.

Proof: If we split $[n]$ into k nonempty intervals, then, by the product formula, the number of ways to carry out the first task on each interval is given by the generating function $A(x)^k$. Then there are b_k ways to carry out the second task on the set of the given k intervals, leading to the generating function $b_k A(x)^k$. Summing over all k, we get that

$$C(x) = \sum_{k \geq 0} b_k A(x)^k = B(A(x)).$$

$\diamondsuit$

Example 3.29 *A soccer coach has her n players stand in a line. Then she breaks the line at a few places, to form nonempty units, and chooses a leader in each unit. Finally, she chooses one of the units for a specific task. Find the generating function for the number c_n of ways she can proceed.*

Solution: Clearly, there are m ways to choose a leader in a unit of m players, so $a_m = m$ for $m > 0$. Similarly, there are k ways to choose a unit out of k units, so $b_k = k$ for $k \geq 1$, with $b_0 = 1$. Therefore, we have $A(x) = \sum_{m \geq 1} mx^m = \frac{x}{(1-x)^2}$, and $B(x) = 1 + \sum_{k \geq 1} kx^k = 1 + \frac{x}{(1-x)^2}$. So Theorem