(D) Compositional formulae

1. Ordinary generating functions:
   (a) If $A(x)$ counts the ways to carry out a certain task on nonempty sets, and $B(x)$ counts the ways to split $[n]$ into nonempty and disjoint subintervals, and then carry out the task counted by $A(x)$ on each of these subintervals, then
   $$B(x) = \frac{1}{1 - A(x)}.$$
   (b) If $A(x)$ counts the ways to carry out a certain task on nonempty sets, $B(x)$ counts the ways to carry out another task on nonempty sets, and $C(x)$ counts the ways to split $[n]$ into nonempty subintervals, then to carry out the first task on each interval and the second task at the set of the intervals, then
   $$C(x) = B(A(x)).$$
2. Exponential generating functions:
   (a) Exponential formula. If $A(x)$ counts the ways to carry out a certain task on nonempty sets, and $B(x)$ counts the ways to partition $[n]$ into nonempty and disjoint blocks, and then carry out the task counted by $A(x)$ on each of these blocks, then
   $$B(x) = \exp A(x).$$
   (b) Compositional formula. If $A(x)$ counts the ways to carry out a certain task on nonempty sets, and $B(x)$ counts the ways to carry out a second task on nonempty sets, and $H(x)$ counts the ways to partition $[n]$ into nonempty blocks, then carry out the task counted by $A(x)$ on each of these blocks, and then carry out the task counted by $B(x)$ on the set of blocks, then
   $$H(x) = B(A(x)).$$

3.8 Exercises

1. Prove that, for all positive integers k,
$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n.$$

2. How can we quickly verify that formula (3.7) is indeed correct for all n?
3. Find an explicit formula for a_n if $a_0 = 2$ and if, for all integers $n \geq 1$, we have $a_n = 4a_{n-1} - 3$.
4. Find an explicit formula for a_n if $a_0 = 0$, and $a_1 = 1$, and if, for all integers $n \geq 2$, we have $a_n = 4a_{n-1} - 4a_{n-2}$.
5. Find an explicit formula for a_n if $a_0 = 0$, and $a_1 = 1$, and if, for all integers $n \geq 2$, we have $a_n = 4a_{n-1} - 5a_{n-2}$.
6. (Basic knowledge of linear algebra required.) Let V be the set of all sequences of complex numbers $\{a_n\}$ that satisfy the recurrence relation

$$a_n = pa_{n-1} + qa_{n-2} \tag{3.34}$$

for all $n \geq 2$, where p and q are fixed real numbers.

   (a) Prove that V is a vector space over the field of real numbers.
   (b) What is the dimension of V?

7. Let V be defined as in the previous exercise. Find all elements $\{a_n\}$ of V whose terms are of the form $a_n = a^n$ for all nonnegative integers n.
8. + Let V be defined as in Exercise 6. Find a basis for V, including explicit formulae for the elements of the basis you found.
9. Solve the recurrence relation $a_0 = 1$ and $a_n = 3\sum_{i=0}^{n-1} a_i$ for all $n \geq 1$.
10. Let $F_1, F_2, \cdots$ be an infinite sequence of formal power series satisfying $F_i(0) = 1$ for all i. Prove that, if there exists a positive integer n so that infinitely many elements of the sequence contain an x^n-term (with a nonzero coefficient), then the infinite product $\prod_{i\geq 1} F_i$ is not defined.
11. Is the converse of the statement of the previous exercise true?
12. Let $p_{odd}(n)$ be the number of partitions of the integer n into odd parts, with $p_{odd}(0) = 1$. Find the ordinary generating function of the numbers $p_{odd}(n)$.
13. Let $p_d(n)$ be the number of partitions of the integer n into distinct parts, with $p_d(0) = 1$. Find the ordinary generating function of the numbers $p_d(n)$.
14. + Compare the results of the previous two exercises. What do they tell us about the connection between $p_{odd}(n)$ and $p_d(n)$?
15. + Express the infinite product

$$\prod_{i\geq 1}(1 - x^i)$$

as an infinite sum, using only one summation sign.

16. Use the result of the previous exercise to prove a recurrence relation for the numbers $p(n)$.
17. Find an explicit formula for a_n if $a_n = na_{n-1} + (n+1)!$ for $n \geq 1$, and $a_0 = 0$.
18. Find an explicit formula for a_n if $a_n = na_{n-1} + (-1)^n$ and $a_0 = 1$.
19. Find a combinatorial proof for the result of Example 3.22. Do not use induction.
20. Let b_n be defined as in Example 3.23. Prove that

$$b_n = 2(b_{n-1} + b_{n-2} + \cdots + b_0).$$

Then deduce from this that $b_n = 2 \cdot 3^{n-2}$ for $n \geq 2$. Do not use generating functions.

21. Let $OS(n,3)$ be the number of *ordered partitions* of $[n]$ into three nonempty blocks. That is, $OS(n,3) = 3!S(n,3)$ since now we can arrange the blocks of any partition counted by $S(n,3)$ in six different ways.

    (a) Find an explicit formula for the exponential generating function

$$OS_3(x) = \sum_{n \geq 0} OS(n,3) \frac{x^n}{n!}.$$

    (b) Deduce a formula for the numbers $OS(n,3)$ and the numbers $S(n,3)$.

In the following three exercises, the reader may want to translate the problem into the language of lattice paths.

22. + Two grandmasters played a series of chess games using the following scoring system: If a game has a winner, the winner gets one point and the loser gets no points. If a game is a tie, both players get one point. Their series ended in a tie of $n - n$. At the end, a journalist noticed that tie games only occurred when the current aggregate score was a tie. Let d_n be the number of possible ways this could happen, and set $d_0 = 1$.

    Find the ordinary generating function $D(x) = \sum_{n \geq 0} d_n x^n$.

23. + Two grandmasters play a series of n chess games using the traditional scoring system. That is, a win is one point, a tie is a half point for both players, and a loss is zero points. Let M_n be the number of ways the series can end up in an aggregate tie (that is, a tie of $\frac{n}{2} - \frac{n}{2}$) if grandmaster A never trails. Find the ordinary generating function of the numbers M_n.