24. + Let us modify the scoring system of the previous exercise so that, in case of a tie, both players get one point. The rest of the conditions are unchanged. Let r_n be the number of ways the series can now end in an aggregate tie of $n-n$, and set $r_n = 1$. Find the ordinary generating function of the numbers r_n.

25. Let $f(n)$ be the number of ways we can tile a $1 \times n$ rectangle using red and blue tiles of size 1×1, and black, white, and yellow tiles of size 1×2. Find the ordinary generating function for the numbers $f(n)$, then find an explicit formula for these numbers.

26. Let $r \geq 2$ be a fixed positive integer, and let $h_r(n)$ be the number of compositions of n into parts that are of the form $rk+1$. Find the closed form for the generating function $H_r(x) = \sum_{n \geq 0} h_r(n)x^n$. Explain the result in the special case of $r = 2$.

27. Let $d \geq 2$ be a fixed positive integer. Find the ordinary generating function for the number of partitions of the integer n in which each part occurs less than d times.

28. Let $d \geq 2$ be a fixed positive integer. Find the ordinary generating function for the number of partitions of the integer n in which no part is divisible by d.

29. Compare the results of the previous two exercises. What is your conclusion?

30. We deduced formula (3.19) for the generating function of the Catalan numbers using the Product formula. Deduce (3.19) using the Compositional formula.

3.9 Solutions to exercises

1. Expanding the left-hand side by the binomial theorem, we see that the coefficient of x^n is $(-1)^n\binom{-k}{n}$. If we can show that this is equal to $\binom{n+k-1}{k-1}$, we will be done. We have

$$\begin{aligned}(-1)^n\binom{-k}{n} &= (-1)^n \cdot \frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}\\ &= (-1)^{2n} \cdot \frac{k(k+1)\cdots(n+k-1)}{n!} = \binom{n+k-1}{n}\\ &= \binom{n+k-1}{k-1}.\end{aligned}$$

2. For $n = 0$, the formula is correct as it gives $a_0 = 5$. Now assume the formula is correct for n. Recall that the sequence $\{a_n\}$ is defined by

$a_{n+1} = 3a_n - 1$ for $n \geq 0$. Therefore, the fact that (3.7) is correct for n implies

$$a_{n+1} = 3\left(5 \cdot 3^n - \frac{3^n - 1}{2}\right) - 1 = 5 \cdot 3^{n+1} - \frac{3^{n+1} - 1}{2},$$

so (3.7) is correct for $n+1$ as well. So, by induction, (3.7) is correct for all nonnegative integers n.

3. Let $A(x)$ be the ordinary generating function of the sequence. Multiply both sides of the recursion by x^n, then sum over $n \geq 1$ to get

$$A(x) - 2 = 4xA(x) - \frac{3x}{1-x}.$$

This yields

$$\begin{aligned} A(x) &= \frac{2}{1-4x} - \frac{3x}{(1-x)(1-4x)} = \frac{2}{1-4x} - \frac{1}{1-4x} + \frac{1}{1-x} \\ &= \frac{1}{1-4x} + \frac{1}{1-x} \\ &= \sum_{n\geq 0} 4^n x^n + \sum_{n \geq 0} x^n. \end{aligned}$$

Therefore, the coefficient of x^n in $A(x)$ is $a_n = 4^n + 1$.

4. Start as in Example 3.7. If $A(x)$ is the ordinary generating function of the sequence, then the usual steps will lead us to the functional equation

$$A(x) - x = 4xA(x) - 4x^2 A(x). \tag{3.35}$$

Therefore, we get

$$\begin{aligned} A(x) &= \frac{x}{1-4x+4x^2} = \frac{x}{(1-2x)^2} \\ &= x\sum_{n\geq 0}\binom{-2}{n}(-2x)^n = x\sum_{n\geq 0}\binom{n+1}{n}(-1)^n(-2)^n x^n \\ &= x\sum_{n\geq 0}(n+1)\cdot 2^n x^n \sum_{n\geq 0}(n+1)\cdot 2^n x^{n+1} = \sum_{n\geq 1} n\cdot 2^{n-1}x^n. \end{aligned}$$

Therefore, we have $a_n = n \cdot 2^{n-1}$.

5. Proceeding as in both Example 3.7 and the previous exercise, we get the equation

$$A(x) - x = 4xA(x) - 5x^2 A(x),$$

which looks similar to (3.35), but is actually quite different from it, as we will see. Expressing $A(x)$ from the previous equation, we get

$$A(x) = \frac{x}{1-4x+5x^2}.$$

In this case, it is somewhat harder to find the partial fraction decomposition of the right-hand side, since the denominator has *complex roots*, namely, $0.4 \pm 0.2i$. Set $\alpha = 0.4 + 0.2i$ and $\beta = 0.4 - 0.2i$. Then, after simplifying by 5, we are looking for numbers C and D so that

$$\frac{C}{x-\alpha} + \frac{D}{x-\beta} = \frac{0.2x}{0.2(1-4x+5x^2)},$$

$$C(x-\beta) + D(x-\alpha) = 0.2x.$$

This leads to $C + D = 0.2$ and $C\beta + D\alpha = 0$. Solving this system, we get $C = 0.1 - 0.2i$ and $D = 0.1 + 0.2i$. Therefore, we have

$$\begin{aligned} A(x) &= \frac{0.1-0.2i}{x-\alpha} + \frac{0.1+0.2i}{x-\beta} = -\frac{0.1-0.2i}{\alpha(1-\frac{x}{\alpha})} - \frac{0.1+0.2i}{\beta(1-\frac{x}{\beta})} \\ &= -\frac{0.1-0.2i}{\alpha}\sum_{n\geq 0}\frac{x^n}{\alpha^n} - \frac{0.1+0.2i}{\beta}\sum_{n\geq 0}\frac{x^n}{\beta^n} \\ &= \frac{i}{2}\sum_{n\geq 0}\frac{x^n}{\alpha^n} - \frac{i}{2}\sum_{n\geq 0}\frac{x^n}{\beta^n}. \end{aligned}$$

We now find a_n as the coefficient of x^n in a_n, that is,

$$a_n = \frac{i}{2}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right).$$

There are several ways to bring this equation to a nicer form. To start, note that $1/\alpha = 2 + i$ and $1/\beta = 2 - i$, so we have

$$a_n = \frac{i}{2}\left((2+i)^n - (2-i)^n\right).$$

If we expand $(2+i)^n$ and $(2-i)^n$ by the binomial theorem, and then we compute their difference, we see that the terms in which the exponents of i are even will cancel. Therefore, we get

$$\begin{aligned} a_n &= \frac{i}{2}\left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n} 2\binom{n}{j}2^{n-j}i^j\right) \\ &= \sum_{\substack{j=1 \\ j \text{ odd}}}^{n}\binom{n}{j}2^{n-j}i^{j-1} \\ &= n\cdot 2^{n-1} - \binom{n}{3}2^{n-3} + \binom{n}{5}2^{n-5} - \cdots. \end{aligned}$$

6. (a) These sequences form a vector space, since, if $\{a_n\}$ and $\{b_n\}$ both satisfy the recurrence relation (3.34), then so does their sum, that is, the sequence $\{a_n + b_n\}$. Similarly, if $\{a_n\}$ satisfies (3.34), then so does its constant multiple, that is, the sequence $\{ca_n\}$ for any $c \in \mathbf{R}$.

(b) The dimension of this vector space is two. Indeed, let $\{a_n\}$ be the sequence in V for which $a_0 = 1$ and $a_1 = 0$, and let $\{b_n\}$ be the sequence in V for which $b_0 = 0$ and $b_1 = 1$. These terms uniquely determine these sequences. Then $\{a_n\}$ and $\{b_n\}$ are linearly independent. Furthermore, any other sequence in V can be obtained as their linear combination. Indeed, if $\{c_n\}$ is a sequence in V with initial terms c_0 and c_1, then we have $\{c_n\} = c_0\{a_n\} + c_1\{b_n\}$.

7. Let us assume that $a_n = a^n$. Then (3.34) turns into

$$a^n = pa^{n-1} + qa^{n-2}.$$

Excluding the uninteresting possibility of $a = 0$, this is equivalent to

$$a^2 - pa - q = 0. \tag{3.36}$$

That is, $a_n = a^n$ will be a solution for (3.34) if and only if a is a solution of (3.36).

8. We have seen in the solution of the previous exercise that we must look for the solutions of (3.36). If it has two distinct solutions (real or complex), say a_1 and a_2, then the sequences $\{a_1^n\}$ and $\{a_2^n\}$ are two linearly independent elements of V. We have seen in the solution of Exercise 6 that $\dim V = 2$, so it follows that $\{a_1^n\}$ and $\{a_2^n\}$ form a basis of V.

If (3.36) has repeated real root a, then we know that $\{a^n\}$ is a solution. To find a second linearly independent solution, note that a being a repeated root implies that

$$a^2 - pa - q = (a - p/2)^2 = 0,$$

so $a = p/2$ and $q = -p^2/4$. In this case, a routine computation shows that $\{na^n\}$ is a solution, and this solution is linearly independent of the solution $\{a^n\}$.

While the reader may be asking how one is supposed to guess this, the reader has probably taken a course in differential equations, where the very same idea is used when the characteristic equation of a second-order equation with constant coefficients has repeated real roots.

9. Let $A(x)$ be the generating function of the sequence we are looking for. Multiply both sides of the defining equation by x^n, and sum over all $n \geq 1$. We get

$$\begin{aligned} A(x) - 1 &= 3\sum_{n\geq 1}\left(\sum_{i=0}^{n-1} a_i\right) x^n, \\ A(x) - 1 &= 3\frac{xA(x)}{1-x}. \end{aligned}$$

In the last step, we used the fact that $\sum_{i=0}^{n-1}$ is the coefficient of x^{n-1} in $A(x)/(1-x)$. Solving for $A(x)$,

$$\begin{aligned} A(x) &= \frac{1-x}{1-4x} = \frac{1}{1-4x} - \frac{x}{1-4x} \\ &= \sum_{n\geq 0} 4^n x^n - \sum_{n\geq 0} 4^{n-1} x^n = \sum_{n\geq 0} 3\cdot 4^{n-1} x^n. \end{aligned}$$

Therefore, $a_n = 3\cdot 4^{n-1}$ for $n \geq 1$.

Alternatively, we could have written up the defining recurrence relation for a_{n+1}, and then subtracted the recurrence relation for a_n from it, to get the formula $a_{n+1} - a_n = 3a_n$, or $a_{n+1} = 4a_n$ for $n \geq 1$, with $a_0 = 1$ and $a_1 = 3$, and then solved that recurrence relation.

10. What would $[x^n]\prod_{i\geq 1} F_i(x)$ be?

11. Yes. If there is no such n, then for every m only a finite number of the power series contain an x^n-term for all $n \leq m$, so $[x^m]\prod_{i\geq 1} F_i(x)$ is finite.

12. Using the same technique as in Example 3.17, we get

$$\sum_{n\geq 0} p_{odd}(n)x^n = \prod_{\substack{i\geq 1 \\ i\ odd}} \frac{1}{1-x^i}. \tag{3.37}$$

13. Following the discussion after Example 3.17, we see that we have

$$\sum_{n\geq 0} p_d(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{j\geq 1}(1+x^j). \tag{3.38}$$

14. We claim that $p_{odd}(n) = p_d(n)$ for all n. We can prove this by showing that the two sequences have identical generating functions, that is, by showing that $\sum_{n\geq 0} p_{odd}(n)x^n = \sum_{n\geq 0} p_d(n)x^n$. By the results of the two previous exercises, this is equivalent to showing that

$$\prod_{\substack{i\geq 1 \\ i\ odd}} \frac{1}{(1-x^i)} = \prod_{j\geq 1}(1+x^j). \tag{3.39}$$

In order to prove the last inequality, let us multiply both sides by $\prod_{j\geq 1}(1-x^j)$. Then on the left-hand side we will have

$$\frac{\prod_{j\geq 1}(1-x^j)}{\prod_{\substack{i\geq 1 \\ i\ odd}}(1-x^i)} = \prod_{\substack{i\geq 1 \\ i\ even}} (1-x^i),$$

whereas on the right-hand side we will have

$$\prod_{i\geq 1}(1-x^j)(1+x^j) = \prod_{i\geq 1}(1-x^{2i}).$$

So the two sides of (3.39) are indeed equal as claimed.

15. The infinite product we are evaluating is quite similar to the generating function $\sum_{n\geq 0} p_d(n)x^n$ that we computed in the solution of Exercise 13. The difference lies in the *signs*. That is, a partition of n that consists of an *odd* number of distinct parts will contribute $-x^n$ to the generating function, and a partition of n that consists of an *even* number of distinct parts will contribute x^n to the generating function. By Euler's pentagonal number theorem (Theorem 2.28), these terms will cancel each other except when n is pentagonal. See that theorem for the precise details. Those details yield that

$$\begin{aligned} A(x) = \prod_{i\geq 1}(1-x^i) &= \sum_{-\infty}^{\infty}(-1)^n x^{n(3n+1)/2} \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} + \cdots . \end{aligned}$$

16. Recall the identity $B(x) = \sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1}\frac{1}{1-x^i}$. Comparing this with the result of the previous exercise, we see that $A(x)B(x) = 1$. This implies that $[x^n](A(x)B(x)) = 0$ if $n \geq 1$. Expressing this by the coefficients of $A(x)$ and $B(x)$, we get the identity

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots = 0,$$

$$p(n) = \sum_{k=1}^{n}(-1)^{k+1}p\left(\frac{k(3k+1)}{2}\right) + \sum_{k=1}^{n}(-1)^{k+1}p\left(\frac{k(3k-1)}{2}\right),$$

where $p(m) = 0$ if $m < 0$.

17. Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ be the exponential generating function of the sequence a_n. Multiply both sides of the defining equation by $\frac{x^n}{n!}$ and sum over all $n \geq 1$ to get

$$\sum_{n\geq 1} a_n \frac{x^n}{n!} = \sum_{n\geq 1} n a_{n-1}\frac{x^n}{n!} + \sum_{n\geq 1}(n+1)x^n.$$

It is easy to recognize $A(x)$ on both sides. Note that the last term on the right-hand side is simply $\left(\frac{1}{1-x}\right)' - 1$. Therefore, the previous equation leads to

$$A(x) = xA(x) + \frac{1}{(1-x)^2} - 1,$$

$$A(x)(1-x) = \frac{1}{(1-x)^2} - 1,$$

$$A(x) = \frac{1}{(1-x)^3} - \frac{1}{1-x}.$$

All that is left to do is to find the coefficient of x^n on the right-hand side. In the second term of the right-hand side, it is 1. The first term is equal to

$$\frac{1}{2}\left(\frac{1}{1-x}\right)'' = \frac{1}{2}\left(\sum_{n\geq 0} x^n\right)'' = \frac{1}{2}\sum_{n\geq 2} n(n-1)x^{n-2}.$$

So the coefficient of x^n in the first term of the right-hand side is $\binom{n+2}{2}$, bringing the coefficient of x^n on the right-hand side to $\binom{n+2}{2} - 1$. Therefore,

$$a_n = n!\left(\binom{n+2}{2} - 1\right) = \frac{(n+2)!}{2} - n!.$$

18. Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ be the exponential generating function of the sequence. Multiplying both sides by $x^n/n!$, and summing over $n \geq 1$, we get

$$\sum_{n\geq 1} a_n \frac{x^n}{n!} = \sum_{n\geq 1} a_{n-1}\frac{x^n}{(n-1)!} + \sum_{n\geq 1}(-1)^n \frac{x^n}{n!},$$

$$A(x) - 1 = xA(x) + e^{-x} - 1.$$

This leads to

$$\begin{aligned} A(x) &= \frac{e^{-x}}{(1-x)} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) x^n. \end{aligned}$$

Therefore, the coefficient of $x^n/n!$ in $A(x)$ for $n \geq 1$ is

$$a_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

19. First, choose the president of the conference committee in one of n ways. The remaining $n-1$ people can be sent to any of the three committees as long as the appropriations committee has an even number of members and the business committee has an odd number of members.

Let us look at the case of even n first. If n is even, then $n-1$ is odd. Let a, b, and c denote the number of members of the three committees. First, we claim that exactly half of the cases in which $a+b$ is odd are good. Indeed, in this case there is a bijection between the set of good assignments and bad assignments. This bijection simply swaps all senators from appropriations to business and vice versa. So if a were odd and b were even, and the assignment was therefore bad, then our bijection turns into one in which a is even and b is odd, which is a good assignment. The converse is also true.

Therefore, our task is now reduced to proving that there are exactly $(3^{n-1}+1)/2$ words of length $n-1$ consisting of letters A, B, and C in which $a+b$ is odd. And that is true, since, if w is such a word, with c letters C, then there are $\binom{n-1}{c}2^{n-1-c}$ possibilities for c. Summing over all even c, we get

$$\sum_{\substack{c=0 \\ c \quad even}}^{n-2} \binom{n-1}{c} 2^{n-1-c} = \frac{(2+1)^{n-1}+(2-1)^n}{2}.$$

The case of odd n is very similar and is therefore left to the reader.

20. Consider an allowed arrangement of an n-page thesis, and remove the last page. Then there are two possibilities:

    (a) The last chapter is still properly arranged. Then there are b_{n-1} possibilities for the remaining part of the thesis and two possibilities for the removed page, resulting in $2b_{n-1}$ possibilities for the original thesis.

    (b) The last chapter is no longer properly arranged. That means that the last chapter no longer contains pages of both types. Let us assume that the last chapter consisted of i pages, then we have b_{n-i} possibilities for the chapters other than the last one and two possibilities for the last chapter (all pages are of one kind, the last page is the other kind). This yields $2\sum_{j=0}^{n-2} b_j$ more possibilities.

    Now that our recurrence relation is proved; it is routine to prove that $b_n = 2\cdot 3^{n-2}$ by induction.

21. (a) We simply have to split $[n]$ into three nonempty blocks, then carry out the trivial task on each block. The trivial task can be carried out in exactly one way. Therefore, we have

$$A(x) = B(x) = C(x) = \sum_{n\geq 1}\frac{x^n}{n!} = e^x - 1.$$

So the product formula implies that the exponential generating function for the number of ways to carry out the composite task is

$$\begin{aligned} S_3(x) &= A(x)B(x)C(x) \\ &= (e^x-1)^3 \\ &= e^{3x} - 3e^{2x} + 3e^x - 1. \end{aligned}$$

(b) It follows from the result of the previous exercise and the identity $e^{kx} = \sum_{n\geq 0} k^n \frac{x^n}{n!}$ that

$$OS(n,3) = 3^n - 3\cdot 2^n + 3$$

for positive n. Therefore, $S(n,3) = (3^{n-1} - 2^n + 1)/2$ for positive n, agreeing with our earlier results.

22. We can visualize the question as follows: The number d_n is the number of lattice paths from $(0,0)$ to (n,n) using steps $(1,0)$, $(0,1)$, and, when on the main diagonal, $(1,1)$.

    Let p be such a lattice path that does not start with a $(1,1)$ step. The number of such paths is $d_n - d_{n-1}$, and their generating function is $D(x)(1-x)$. Such paths can be decomposed into two parts, the part before they first touch the main diagonal and the part after that. The first parts are enumerated by the generating function $2xC(x)$, where $C(x)$ is the generating function of the Catalan numbers that we computed in Example 3.16. The reader may want to read that example again for further justification of the last sentence. Indeed, the first part of these paths is a northeastern lattice path that is either below the diagonal or above it, explaining the factor 2. The second parts are enumerated by $D(x)$ itself. Using the product formula, we get

    $$D(x)(1-x) - 1 = 2xC(x)D(x),$$

    since this decomposition is only possible if $n \geq 1$. Expressing $D(x)$ from this, we get

    $$D(x) = \frac{1}{1-x-2xC(x)} = \frac{1}{\sqrt{1-4x}-x} = \frac{\sqrt{1-4x}+x}{1-4x-x^2}.$$

23. In the language of lattice paths, M_n is the number of lattice paths from $(0,0)$ to $(n/2, n/2)$ with steps $(1,0)$, $(0,1)$, and $(1/2,1/2)$ that never go above the diagonal $x = y$. Otherwise, this proof will be very similar to that of Example 3.16. The reader may want to reread that proof first.

    To start with, we have $M_0 = 1$. Let $M(x) = \sum_{n\geq 0} M_n x^n$. Let us look at all series of $n \geq 1$ games ending in an aggregate tie that did not start with a tie game. Their number is $M_n - M_{n-1}$ for $n \geq 1$, and therefore their generating function is $M(x)(1-x) - 1$. We will now obtain this same generating function in another way, using the product formula. Let us call the first moment in which the score is a tie (after 0-0) the *critical moment*. The critical moment breaks our series of games into two parts, namely, the part before the critical moment and the part after the critical moment. The part after the critical moment is just a series like the original one, so the generating function of the relevant numbers is $M(x)$. The part before the critical moment is a series that is never a tie except at the beginning and at the end. By an argument analogous to that of Example 3.16, we see that the generating function of the relevant numbers is $M(x)x^2$ (the series starts with a win of A, and the critical moment arrives with a loss of A, so there are M_{i-2} choices for this part, where i is the length of the part). By the product formula, this leads to the functional equation

$$M(x)(1-x) - 1 = M(x)x^2 \cdot M(x),$$

$$M(x)^2x^2 + M(x)(x-1) + 1 = 0.$$

Solving this quadratic equation for $M(x)$, and verifying that the *negative* root is the combinatorially meaningful one, we get

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

The numbers M_n are called the *Motzkin numbers*. See [72] for more than a dozen interpretations of these numbers. Some additional interpretations can be found in [13].

24. In the language of lattice paths, r_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$, and $(1,1)$ that never go above the main diagonal. Otherwise, this exercise is similar to the previous exercise, with some significant differences. Let $R(x)$ be the generating function we are looking for. Then $R(x)(1-x)-1$ is the generating function for the numbers of possible series that do not start with a tie game. Define the critical moment of a series as in the solution of the previous exercise. Then the product formula shows that

$$R(x)(1-x) - 1 = xR(x) \cdot R(x),$$

since the part before the critical moment must start with a win of A, end with a win of B, and contain a series counted by r_{i-1} in between, where the critical moment arrives when the score is $i-i$. Solving the last equation for $R(x)$, we get

$$R(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}. \tag{3.40}$$

The numbers r_n are called the *Schröder numbers*. See the remark at the end of the solution of the previous exercise to find out where to read more about these numbers.

25. We will use the compositional formula of ordinary generating functions. We partition the interval $[1,n]$ into an unspecified number of

intervals, then on each interval we carry out the task of tiling that interval *using one tile only*. There are two ways to do that for intervals of size 1, three ways to do it for intervals of size 2, and no ways to do it for larger intervals. Therefore, $A(x) = 2x + 3x^2$. Hence the compositional formula yields $F(x) = 1/(1-A(x)) = 1/(1-2x-3x^2)$. This leads to the formula $f(n) = \left(3^{n+1} + (-1)^n\right)/4$.

26. We will use the compositional formula of ordinary generating functions. We partition the interval $[1, n]$ into an unspecified number of intervals, then on each interval we carry out the task of partitioning that interval into one part of size $rk + 1$, for some nonnegative integer k. This is possible in one way if that interval had length $rk+1$, and in zero ways otherwise. Therefore, the inside generating function is $A(x) = x + x^{r+1} + x^{2r+1} + \cdots = x/(1 - x^r)$, so the compositional formula yields

$$H(x) = \frac{1}{1 - A(x)} = \frac{1 - x^r}{1 - x - x^r}.$$

In the special case of $r = 2$, we get

$$\frac{1 - x^2}{1 - x - x^2} = 1 + \frac{x}{1 - x - x^2},$$

which is the generating function of the Fibonacci numbers (shifted by 1).

27. Let $A_d(x)$ be the requested generating function. Then we have

$$A_d(x) = \prod_{i \geq 1} \left(1 + x^i + \cdots + x^{(d-1)i}\right) = \prod_{i \geq 1} \frac{1 - x^{di}}{1 - x^i}.$$

28. Let $B_d(x)$ be the requested generating function. Then we have

$$B_d(x) = \prod_{\substack{j \geq 1 \\ j \neq dk}} \frac{1}{1 - x^j}.$$

29. The equality $A_d(x) = B_d(x)$ holds as the factors $1-x^j$ where $j = di$ cancel out in $A_d(x)$. Therefore, for all n and $d \geq 2$, the number of partitions of n in which each part occurs less than d times is equal to the number of partitions of n in which no part is divisible by d.

30. The moments at which the game was tied break up the sequence of goals into an undetermined number of segments. Each such segment starts with the home team scoring, and ends with the visiting team scoring. In between, there is a regular sequence of goals in