p_{j+1} is equal to the number of permutations p for which $i(p) = k$ and $p_{j-1} < p_j > p_{j+1}$.

3. Let n be a positive integer for which $\binom{n}{2}$ is even. Prove that the number $b(n, \binom{n}{2}/2)$ is even.

4.5 Advanced applications of generating functions to permutation enumeration

Let us consider a few counting problems where generating functions are still very useful, but we have to use them in ways that we have not seen before.

4.5.1 The combinatorial meaning of the derivative

In Theorem 4.21, we proved the interesting formula

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1) \tag{4.14}$$

for any fixed positive integer n.

Now let us take the *derivative* of both sides with respect to x. We get the identity

$$\sum_{k=1}^{n} k \cdot c(n,k)x^{k-1} = \sum_{i=0}^{n-1} \frac{x(x+1)\cdots(x+n-1)}{x+i}.$$

Note that, differentiating the right-hand side of (4.14), we have simply used the product rule, which was easy to do since the derivative of any individual factor $(x+i)$ is 1.

Now let us substitute $x = 1$ in the last displayed equation, to get

$$\sum_{k=1}^{n} k \cdot c(n,k) = \sum_{j=1}^{n} \frac{n!}{j}. \tag{4.15}$$

The crucial observation is that the expression on the left-hand side, $\sum_{k=1}^{n} k \cdot c(n,k)$, is simply the number of *all* cycles in all permutations of length n. Indeed, permutations counted by $c(n,k)$ contribute k cycles to that total. So formula (4.15) is in fact a formula for the total number of cycles in all permutations of length n. Equivalently, if we divide both sides of (4.15) by $n!$, we get that the *average* number of cycles in permutations of length n is $\sum_{j=1}^{n} \frac{1}{n}$. This number is often called the nth *harmonic number*. While this result can be proved without using generating functions (see Exercise 18), the idea of taking the derivative of a generating function and then finding its value at $x = 1$ is often useful in similar situations.