4.8 Exercises

1. Prove that, for all positive integers k and n satisfying $k \leq n$, we have $A(n, k) = A(n, n+1-k)$. Try to give two different solutions.
2. Let $S \subseteq [n-1]$, and let $\alpha(S)$ denote the number of n-permutations whose descent set is *contained* in S.

   Find the one-element set $\{i\} \subseteq [n-1]$ for which $\alpha(\{i\})$ is maximal.
3. Let $n = 10$ and let $S = [3, 7]$. Compute $\alpha(\{3, 7\})$.
4. Let $S = \{i_1, i_2, \cdots, i_k\} \subseteq [n-1]$. Find an explicit formula for $\alpha(S)$.
5. How many 10-permutations are there whose set of descents is *equal* to the set $\{3, 7\}$?
6. Let $S = \{i_1, i_2, \cdots, i_k\} \subseteq [n-1]$. Let $\beta(S)$ denote the number of n-permutations whose set of descents is *equal* to S. Find a formula for $\beta(S)$. Your answer may contain the α function.
7. Let

   $$M(n, k) = \sum_{k=1}^{n} A(n, k).$$

   That is, $M(n, k)$ is the number of n-permutations with at most k ascending runs. Prove, by constructing an appropriate surjection, that

   $$M(n, k) \leq k! S(n, k).$$

8. The *order* of the permutation p is the smallest positive integer k for which $p^k = id$.

   (a) Prove that, if p and q have the same type, then their order is the same.

   (b) How can we figure out that order given just the type of p and q?
9. Let S be the infinite matrix whose rows are indexed by the nonnegative integers and whose row k consists of the Stirling numbers $S(n, 0), S(n, 1), \cdots$. Let s be the similarly defined infinite matrix, but with the numbers $s(n, k)$ replacing the numbers $S(n, k)$. Find a matrix identity satisfied by the matrices S and s.
10. An airplane has n seats, and each of them is sold for a particular flight. Passengers board the plane one by one, and the first $n-1$ passengers sit at *random* places, not necessarily their assigned seats. When the last passenger boards the plane, he goes to his assigned seat. If his assigned seat is free, then he takes it. If he finds someone in his assigned seat, then he asks that person to move. Then that person goes to her assigned seat, and acts similarly. This continues until the person currently moving finds her assigned seat empty, and

so nobody else is asked to move. Let $k \in n$. What is the probability that (counting the last passenger) exactly k people had to move because of the arrival of the last passenger?

11. What is the average number of fixed points of all n-permutations?

12. How many 6-permutations are there whose fourth power is the identity permutation?

13. How many 6-permutations are there whose sixth power is the identity permutation?

14. How many 12-permutations are there whose cube contains exactly two 3-cycles?

15. How many even 6-permutations have order four? The order of a permutation is defined in Exercise 8.

16. Find an explicit formula for the exponential generating function for the number of n-permutations whose third power is the identity permutation.

17. Find an explicit formula for the exponential generating function for the number of n-permutations whose order is six.

18. Solve this exercise *without* using generating functions.

    (a) Let $H(n)$ be the average number of cycles of an n-permutation. Prove that

    $$H(n) = \frac{n-1}{n} H(n-1) + \frac{1}{n}(H(n-1)+1).$$

    (b) Find an explicit formula for $H(n)$.

19. Prove that, if $n \geq 1$, then

    $$n! = \sum_{k=0}^{n} \binom{n}{k} D(k),$$

    where $D(k)$ denotes the number of derangements of length k and we set $D(0) = 1$.

20. Prove, by a combinatorial argument, that, for all $n \geq 2$, we have

    $$D(n) = (n-1)(D(n-1) + D(n-2)).$$

21. (a) Find the exponential generating function for the numbers $f_k(n)$ of permutations of length n in which each cycle length is divisible by k.

    (b) Find a formula for the numbers $f_k(n)$ defined in part (a).

    (c) Find a combinatorial proof for the result of part (b).

22. Let p be a prime number. For what $k \in [p]$ will $c(p, k)$ be divisible by p?

23. (Wilson's theorem) Let p be a prime number. Prove that $(p-1)!+1$ is divisible by p.

24. Compute $b(10, 3)$.

25. Prove that, if $n - 1 \geq k$, then

$$b(n, k) = b(n, k-1) + b(n-1, k). \tag{4.19}$$

26. Modify the recursive relation of the previous exercise for the case when $n - 1 < k$.

27. Let $k \leq n - 1$.

    (a) How many n-permutations are there for which $i(p)$ is divisible by k?

    (b) How many n-permutations are there for which $i(p) \equiv r$ for some r, modulo k?

28. Let $Inv(n, k)$ be the set of n-permutations having exactly k inversions. Let

$$a_i = |\{p = p_1 p_2 \cdots p_n \in Inv(n, k) | p_i > p_{i+1}\}|.$$

Prove that a_i is independent of i.

29. Let $k \leq n$ be fixed positive integers. Let K be the multiset that consists of k copies of 1 and $n - k$ copies of 2. Let p be a permutation of K, and define an inversion of p as a pair formed by a 2 and a 1 so that the 2 precedes the 1. For instance, 221 has two inversions, while 1212 has one. We know that altogether, K has $\binom{n}{k}$ permutations, but let us now count them according to their inversions. In particular, define

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \sum_p q^{i(p)}.$$

Here, $i(p)$ is the number of inversions of p, and the sum ranges over all permutations of K. The polynomials $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$ are called *Gaussian coefficients*, or q-binomial coefficients, or *Gaussian polynomials*.

    (a) Compute $\begin{bmatrix} \mathbf{4} \\ \mathbf{2} \end{bmatrix}$, $\begin{bmatrix} \mathbf{5} \\ \mathbf{1} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{5} \\ \mathbf{3} \end{bmatrix}$.

    (b) Prove that $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{n} \\ \mathbf{n-k} \end{bmatrix}$.

30. Prove that

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = q^{n-k} \cdot \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k-1} \end{bmatrix} + \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k} \end{bmatrix}. \tag{4.20}$$

31. Let us enumerate integer partitions whose Ferrers shape fits within an $m \times n$ rectangle according to their sizes. That is, let

$$p(m, n, q) = \sum_{a} q^{|a|},$$

where a ranges over all integer partitions that have at most m rows and at most n columns, and $|a|$ denotes the integer of which a is a partition. For instance, $p(2, 2, q) = 1 + q + 2q^2 + q^3 + q^4$.

Prove that $p(m, n, q) = \left[\begin{smallmatrix}\mathbf{m+n}\\ \mathbf{m}\end{smallmatrix}\right]$.

32. We say that a permutation *avoids* the pattern 132 if it does not have three elements that relate to each other the same way as 1, 3, and 2. That is, if $p = p_1 p_2 \cdots p_n$, then p is 132-avoiding if there are no three indices $i < j < k$ so that $p_j > p_k > p_i$. That is, there are no three entries in p among which the leftmost is the smallest and the one in the middle is the largest, just as in 132. For instance, 42351 is 132-avoiding, but 35241 is not, for the three entries 3, 5, and 4. So we will say that 35241 *contains* 132.

Prove that the number of 132-avoiding n-permutations is the nth Catalan number.

33. Let q be any permutation of length k, and define q-avoiding permutations in an analogous way to 132-avoiding permutations, with q playing the role of 132. Let $S_n(q)$ be the number of n-permutations avoiding the pattern q. For what permutations q does the result of the previous exercise immediately imply that $S_n(q) = C_n$?

34. Prove that, if the n-permutation $p = p_1 p_2 \cdots p_n$ contains a 312-pattern, then it must contain a 312-pattern in which entries playing the role of the entries 3 and 1 of the 312-pattern are *consecutive entries* in p.

35. Find a non-generating function proof for the result of Example 3.37.

36. A *regular tetrahedron* is a solid with four vertices, six edges, and three faces, so that each edge is of the same length. See Figure 4.5 for an illustration.

    (a) Find the number of all symmetries of a regular tetrahedron.

    (b) Find the number of all symmetries of a regular tetrahedron that can be obtained by a series of rotations.

    (c) Symmetries of a regular tetrahedron are permutations of its vertex set. Does the sign (or parity) of such a permutation tell us whether the corresponding symmetry can be obtained by a series of rotations?

37. (+) We have seen how to compute the number of all cycles in all permutations of length n in Section 4.5.1, and also, in Supplementary Exercise 16. Let us consider the following generalization of that