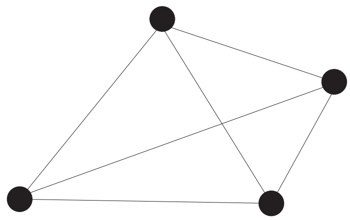

Figure 4.5
A regular tetrahedron.

problem. Take all permutations of length n, and in each of them, label each cycle with a positive divisor of the length of that cycle. Let $h(n)$ be the number of all these labeled cycles. Find the exponential generating function of the sequence of the numbers $h(n)$. The first four values of the sequence $h(n)$ are 1, 3, 11, and 59.

(In other words, $h(n)$ counts ordered triples (p, C, d) so that p is a permutation of length n, while C is a cycle of p, and d is a positive divisor of the length of C.)

4.9 Solutions to exercises

1. Recall that $A(n, k)$ is the number of n-permutations with $k - 1$ descents. If $p = p_1 p_2 \cdots p_n$ has $k - 1$ descents, then its reverse, $p^r = p_n p_{n-1} \cdots p_1$, has $k-1$ ascents, so it has $(n-1)-(k-1) = n-k$ descents. In other words, "taking reverses" is a bijection from the set of permutations counted by $A(n, k)$ into the set of permutations counted by $A(n, n + 1 - k)$.

   Alternatively, instead of taking reverses, we can take *complements*. That is, for $p = p_1 p_2 \cdots p_n$, define p^c as the permutation whose ith entry is $n + 1 - p_i$. If p has $k - 1$ descents, then its complement p^c has $k - 1$ ascents, and the proof is completed as above.

2. In a permutation counted by $\alpha(\{i\})$, the first i entries must form an increasing sequence, and the last $n - i$ entries have to form an increasing sequence. Therefore, once we know the set of the first i entries, we know the permutation. This yields $\alpha(\{i\}) = \binom{n}{i}$. It is easy to prove computationally that for fixed n the maximum of $\binom{n}{i}$ is taken for $i = \lfloor n/2 \rfloor$ (if you do not believe it, see Proposition 6.39).

3. Let $\pi = \{B_1, B_2, B_3\}$ be any ordered partition of [10] in which the

blocks B_1, B_2, B_3 have respective sizes 3, 4, and 3. Let us order the elements of each block increasingly, and write down the elements of the blocks starting with B_1, following with B_2, and then ending with B_3. Clearly, we get a permutation whose descent set is contained in $\{3,7\}$ as each block is ordered increasingly. It is also clear that we obtain each such permutation exactly once. So all we have to do is to find the number of such partitions π. That is not difficult, because we have $\binom{10}{3}$ choices for B_1, then $\binom{7}{4}$ choices for B_2, and because our hands are tied when we choose B_3. Therefore, by the product principle, we have

$$\alpha(\{3,7\}) = \binom{10}{3} \cdot \binom{7}{4}.$$

4. Assume the s_i are listed in increasing order. Using the ideas of the solution of the previous exercise, we get that

$$\begin{aligned}\alpha(S) &= \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\binom{n-s_1-s_2}{s_3-s_2}\cdots\binom{n-s_1-\cdots-s_{k-1}}{s_k-s_{k-1}} \\ &= \binom{n}{s_1, s_2-s_1, \cdots, s_k-s_{k-1}, n-s_k}.\end{aligned}$$

5. In Exercise 3, we have computed the number of permutations of length 10 whose descent set was contained in $\{3,7\}$. That means that the descent set of those permutations was one of the following sets: $\{3,7\}$ or $\{3\}$ or $\{7\}$ or $\emptyset$. Following the line of the previous exercise, for $S \subseteq [10]$ we define $\beta(S)$ to be the number of permutations whose set of descents is equal to S. Then we claim that

$$\beta(\{3,7\}) = \alpha(\{3,7\}) - \alpha(\{3\}) - \alpha(\{7\}) + \alpha(\{\emptyset\}).$$

Indeed, the second and the third terms on the right-hand side subtract the number of permutations whose descent set is *strictly* contained in $\{3,7\}$, and in doing this, they subtract the number of permutations with no descents twice. We have computed $\alpha(\{3,7\})$ in the previous exercise. By the same method, we see that $\alpha(\{3\}) = \binom{10}{3}$, and $\alpha(\{7\}) = \binom{10}{7} = \binom{10}{3}$. As there is only one permutation with no descent, we get that

$$\beta(\{3,7\}) = \binom{10}{3} \cdot \binom{7}{4} - 2 \cdot \binom{10}{3} + 1.$$

6. Using the ideas of the previous exercise, we get

$$\beta(S) = \sum_{T \subseteq S} \alpha(S)(-1)^{|S-T|}.$$

7. Take any of the $S(n,k)$ partitions of $[n]$ into k blocks. Order the elements within each block increasingly, then order the *set* of k blocks in any of $k!$ ways. This way, we get $S(n,k)k!$ permutations, and they all have at most k ascending runs. We will obtain some of them several times; this is why only inequality, not equality, is true.

8. (a) By part (b), the order of a permutation is the least common multiple of all cycle lengths. If two permutations have the same type, then their cycle lengths are pairwise equal, as are their least common multiples.

   (b) If the length of a cycle of a permutation p is t, then for any i, the permutation p^{ti} will contain the entries of that cycle as fixed points. So if m is a multiple of all cycle lengths of p, then $p^m = id$. Therefore, the order of p is the *least common multiple* of all cycle lengths of p.

9. We claim that $Ss = sS = I$, where I is the identity matrix. At this point, the reader may reproach us for the fact that we have not even defined the product of two infinite matrices. Therefore, we will say that we multiply two infinite matrices X and Y very much as if they were finite, that is, the (i,j)-entry of XY is the dot product of row i of X and column j of Y, as long as that dot product is defined. That dot product is the infinite sum $\sum_k x_{ik}y_{kj}$, which is certainly defined if all but a finite number of its summands are equal to 0. If, for some i and j, this dot product is not defined (infinite), then we say that XY is not defined. In our examples, both sS and Ss are defined because $S(n,k) = s(n,k) = 0$ for $k > n$.

   Returning to the proof of our statements, we claim that s and S are inverses of each other. This follows from the fact that the columns of S are in fact the elements of B written in the basis B', and the columns of s are the elements of B' written in the basis B.

   So S is the *change of basis matrix* from B' to B, and s is the change of basis matrix from B to B'. That is, if p is any polynomial in V, and $\mathbf{p_C}$ denotes p written in basis C, then we have

$$\mathbf{p_{B'}} = s\mathbf{p_B},$$

   and

$$\mathbf{p_B} = S\mathbf{p_{B'}}.$$

   This proves that S and s are indeed inverses of each other.

10. The way the n passengers sit down in their assigned seats determines a permutation of $[n]$. A passenger will have to move because of the arrival of the last passenger if and only if she is in the same cycle as the last passenger. Therefore, our task is to determine the probability of a randomly selected cycle of a random permutation being exactly k long.

Since all entries of $[n]$ have the same role in this problem, we might as well answer this question for the cycle containing the maximum entry n. Using the transition lemma, we see that the length of the cycle containing n is k if and only if n is the $(n+1-k)$th entry of $f(p)$. Indeed, n always starts the last cycle of $f(p)$. The probability of n being in any given position is, of course, $1/n$, so that is the probability that the cycle containing n is of length k.

11. Let a_n be the average we are looking for. We claim that $a_n = 1$ for all n, and we will prove this by induction on n.

    For $n = 1$, the statement is obvious. Assume now that $a_{n-1} = 1$. It is clear that there are $(n-1)!$ permutations of length n in which 1 is a fixed point. By induction, the remaining part of the permutation, which is a permutation on the set $\{2, 3, \cdots, n\}$, has one fixed point on average. So permutations in which 1 is a fixed point have two fixed points on average.

    On the other hand, if $k > 1$, then we know from the previous exercise that there are again $(n-1)!$ permutations of length n in which 1 is in a cycle of length k. Elements of this cycle will not be fixed points. Therefore, all fixed points of the permutation will come from the rest of the permutation. The rest of the permutation is a permutation on $n-k$ elements. Therefore, by induction, it has on average one fixed point, *as long as it is not the empty permutation*, that is, as long as $k < n$. If $k = n$, then the remaining part of the permutation has no fixed points.

    Therefore, adding for the three cases—that is, when the length of the cycle containing the entry 1 is one, when it is k, where $1 < k < n$, and when it is n—we get that

$$a_n = \frac{(n-1)! \cdot (1+1) + (n-2)(n-1)! \cdot 1 + (n-1)! \cdot 0}{n!} = \frac{n!}{n!} = 1.$$

12. For a permutation p to satisfy $p^4 = id$, the length of each cycle in p has to be a divisor of 4, that is, 4, 2, or 1. As we are looking at 6-permutations, we have to consider partitions of 6 into parts that are 4, 2, or 1. These are

    - $4+2$, that is, a 4-cycle and a 2-cycle. By Theorem 4.27, there are $\frac{6!}{4 \cdot 2} = 90$ permutations of this type.
    - $4+1+1$. There are $\frac{6!}{4 \cdot 1 \cdot 1 \cdot 2} = 90$ permutations of this type.
    - $2+2+2$. There are $\frac{6!}{2^3 \cdot 3!} = 15$ permutations of this type.
    - $2+2+1+1$. There are $\frac{6!}{2^2 \cdot 2^2} = 45$ permutations of this type.
    - $2+1+1+1+1$. There are $\frac{6!}{2 \cdot 4!} = 15$ permutations of this type.

- $1+1+1+1+1+1$. There is obviously one permutation of this type.

Therefore, there are $90+90+15+45+15+1=256$ permutations of length six whose fourth power is the identity permutation.

We point out, for future reference, that the last four types of permutations are involutions. Therefore, there are 76 involutions of length six.

13. For a permutation p to satisfy $p^6 = id$, the length of each cycle in p has to be a divisor of 6, that is, 6, 3, 2, or 1. As we are looking at 6-permutations, we have to consider partitions of 6 into parts that are 6, 3, 2, or 1. These are

    - 6, that is, when p consists of one 6-cycle. Then by Theorem 4.27, there are $\frac{6!}{6} = 120$ permutations of this type.
    - $3+3$. There are $\frac{6!}{3^2 \cdot 2} = 40$ permutations of this type.
    - $3+2+1$. There are $\frac{6!}{3 \cdot 2} = 120$ permutations of this type.
    - $3+1+1+1$. There are $\frac{6!}{3 \cdot 3!} = 40$ permutations of this type.
    - Involutions. We know from the previous exercise that there are 76 of them.

    Therefore, the number of 6-permutations whose sixth power is the identity is $120+40+120+40+76=396$.

14. What happens to the cycles of p when we take p^3? That depends on the length of the cycle.

    (a) If the length of a cycle was relatively prime to 3, then it will not change. In particular, it will still be relatively prime to 3.

    (b) If, however, the length of the cycle was $3r$, then the cycle will split into three cycles of length r.

    Note that it is only in the second case that p^3 can have a cycle whose length is divisible by three, namely, when r is divisible by three. But then p^3 will immediately have three of those r-cycles. So the number of cycles of p^3 that are of length $3i$ is always a multiple of three. Therefore, it cannot be two.

15. Such permutations must contain a 4-cycle. Since they are even, the rest of their entries must be fixed points. So Theorem 4.27 shows that their number is $\frac{6!}{2! \cdot 4} = 90$.

16. In such permutations, each cycle length is one or three. Therefore, Theorem 4.34 implies that

$$G_C(x) = \exp\left(x + \frac{x^3}{3}\right).$$

17. In such permutations, each cycle length is at 1, 2, 3, or 6. However, either both the cycle lengths 2 and 3 have to occur, or the cycle length 6 has to occur. In other words, we have to exclude permutations whose third or second power is the identity permutation. The only permutation p so that $p^2 = p^3 = id$ is the identity permutation itself. Therefore, by the inclusion-exclusion principle, the generating function we are looking for is

$$\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^6}{6}\right) - \exp\left(x + \frac{x^2}{2}\right)$$
$$- \exp\left(x + \frac{x^2}{2}\right) + \exp x.$$

18. (a) Let p be a permutation of length $n-1$, written in cyclic notation taken not on the set $[n]$, but on the set $\{2, 3, \cdots, n\}$. Now insert 1 into any of the following n positions of p: either to the front of p, forming a new cycle by itself or after any entry x of p, in the same cycle as x. In the first case, the number of cycles grows by one, otherwise it does not change. As the first case occurs in $1/n$ of all times, the formula is proved.

    (b) We claim that
    $$H(n) = \sum_{i=1}^{n} \frac{1}{i}.$$
    This is trivial to prove for $n = 1$, and is routine by induction on n, and by the recursive formula of part (a) after that.

19. Let us write the right-hand side as $\sum_{k=0}^{n} \binom{n}{n-k} D(k)$. Then the right-hand side counts all n-permutations according to their fixed points. Indeed, if an n-permutation has exactly $n-k$ fixed points, then we have $\binom{n}{n-k}$ choices for the set of these fixed points. The permutation of the entries that are not fixed must be a derangement on k entries.

20. We show that the right-hand side also counts all derangements of length n. First, let p be any derangement of length $n-1$. Then the entry n can be inserted into $n-1$ different positions of p, namely, after each entry x in the same cycle as x. This way we get $(n-1)D(n-1)$ derangements, in each of which the entry n is in a cycle of length at least three. So we are still missing those derangements of length n in which the entry n is in a 2-cycle. In these, we have $n-2$ choices for the entry y sharing a cycle with n, and then we have $D(n-2)$ choices for the rest of the permutation. This gives us the missing $(n-2)D(n-2)$ derangements. A little thought shows that we have counted all derangements of length n, so the proof is complete.

21. (a) We use Theorem 4.34, with C being the set of all positive integers divisible by k. Then we get

$$\begin{aligned} G_C(x) &= \exp\left(\sum_{k\geq 1}\frac{x^{kn}}{n}\right) \\ &= \exp\left(\frac{1}{k}\ln(1-x^k)^{-1}\right) \\ &= \left(\frac{1}{1-x^k}\right)^{1/k}. \end{aligned}$$

If the reader wants more details about this computation, she should read Example 4.39 and the paragraphs preceding it. That example is a special case of this exercise, namely, the one with $k=2$.

(b) Again, similarly to Example 4.39, we need to find the coefficient of $x^n/n!$ in $(1-x^k)^{-1/k}$. Obviously, $f_k(n)=0$ if n is not divisible by k. Otherwise, let $n=rk$. We then find that this number is

$$f_k(n) = 1^2 \cdot 2 \cdots (k-1)\cdot(k+1)^2 \cdots (2k-1)\cdot(2k+1)^2 \cdots (rk-1).$$

In other words, $f_k(n)$ is a product of n terms, just as $n!$, but the terms that are divisible by k are missing and are replaced by the integer that is $k-1$ smaller.

(c) This is similar to the proof given in the text for the special case of $k=2$. That is, assume $p=rk$. If p is enumerated by $f_k(n)$, then $p(1)$ cannot be 1, but it can be anything else, that is, we have $rk-1$ choices for $p(1)$. Then $p(p(1))$ can be anything but 1 and $p(1)$, so we have $rk-2$ choices for it. This trend continues until $p^k(1)$. (That is, if $i<k$, then $p^i(1)$ cannot be any of 1 and $p^j(1)$ for $j<i$, yielding $rk-i$ choices.) Then p^i can be 1 as well, giving us $rk-k+1$ choices. Continuing this argument as in the text, the proof follows.

22. We claim that $c(p,k)$ is divisible by p except for $k=1$ and $k=p$. It is obvious that these two cases are indeed exceptions. Otherwise, $c(p,k)$ can be obtained by adding the numbers of all permutations of all types that have k cycles. By Theorem 4.27, all these numbers are fractions in which the numerator is divisible by p and the denominator is not. Therefore, since p is prime, all these numbers are divisible by p, and so is their sum.

23. Recall that $(p-1)!=c(p,1)$. By the result of the previous exercise,

we have that

$$\begin{aligned} p! = \sum_{k=1}^{p} c(p,k) &\equiv 0 \ \ (mod\ p) \\ c(p,1) + c(p,p) &\equiv 0 \ \ (mod\ p) \\ (p-1)! + 1 &\equiv 0 \ \ (mod\ p), \end{aligned}$$

which was to be proved.

24. By Theorem 4.54, we know that $b(n,k)$ is the coefficient of x^3 in $I_{10}(x)$. In other words, $b(n,k)$ is the number of weak compositions of 3 into nine parts, so that the first part is at most 1, and the second part is at most 2. There are altogether $\binom{11}{3}$ weak compositions of 3 into nine parts. One of them violates the condition that the second part is at most 2, and nine of them violate the condition that the first part is at most 1. No weak composition of 3 into nine parts violates both conditions. Therefore,

$$b(10,3) = \binom{11}{3} - 1 - 9 = 165 - 1 - 9 = 155.$$

25. The left-hand side counts all n-permutations with k inversions. The second term of the right-hand side counts those permutations with this property whose last entry is n. Finally, the first term of the right-hand side counts those n-permutations with k inversions in which n is not the last entry. Indeed, in these permutations, n can be interchanged with the entry immediately following it, decreasing the number of inversions by 1, to $k-1$. This way, we will get an n-permutation with $k-1$ inversions in which n is not in the first position; but n could not be there anyway, since that would result in n inversions, which is not allowed.

26. Following the line of thinking of the solution of the previous exercise, we see that we need an additional term on the right-hand side, namely, the number of n-permutations with $k-1$ inversions in which n is in the first position. If n is in the first position, then it is part of $n-1$ inversions, so the rest of the permutation must contain $k-n$ inversions, which can happen in $b(n-1,k-n)$ ways. Therefore, we have

$$b(n,k) = b(n-1,k-n) + b(n,k-1) + b(n-1,k).$$

27. The answer to both questions is $n!/k$. To see this, one only has to consider the term $(1 + x + \cdots + x^{k-1})$ of $I_n(x)$.

28. We claim that $a_i = a_{i+1}$, and it is obvious that then all a_i have to be equal. We can clearly disregard permutations in which both i and $i+1$ are descents, as well as permutations in which neither one is a descent. So all we have to show is that there are as many permutations in $Inv(n,k)$ in which $p_i > p_{i+1} < p_{i+2}$ (set A) as there are permutations in $Inv(n,k)$ in which $p_i < p_{i+1} > p_{i+2}$ (set B). This is easy to do bijectively. Let $p \in A$. Leave the entries on the left of p_i and on the right of p_{i+2} unchanged. Then rearrange the entries p_i, p_{i+1}, p_{i+2} by first reversing them, and then taking complements within this three-element set. [In the terminology of Exercise 32, if the pattern of these three entries was 312, turn it into 231, and if it was 213, turn it into 132.] Because the number of total inversions of the permutation did not change, the new permutation is still in $Inv(n,k)$, and it is obviously in B. The map is bijective, since taking reverse complements is a bijective operation.

29. (a) For $\begin{bmatrix}\mathbf{4}\\\mathbf{2}\end{bmatrix}$, we count the permutations 1122, 1212, 1221, 2112, 2121, and 2211. Therefore,

$$\begin{bmatrix}\mathbf{4}\\\mathbf{2}\end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4.$$

For $\begin{bmatrix}\mathbf{5}\\\mathbf{1}\end{bmatrix}$, we must count the permutations 12222, 21222, 22122, 22212, and 22221. Therefore,

$$\begin{bmatrix}\mathbf{5}\\\mathbf{1}\end{bmatrix} = 1 + q + q^2 + q^3 + q^4.$$

Finally, for $\begin{bmatrix}\mathbf{5}\\\mathbf{3}\end{bmatrix}$, we must count the permutations 11122, 11212, 11221, 12112, 12121, 12211, 21112, 21121, 21211, and 22111. Therefore,

$$\begin{bmatrix}\mathbf{5}\\\mathbf{3}\end{bmatrix} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + 1.$$

(b) Let p be a permutation of the multiset K. Then reverse p, and turn all 1s into 2s, and vice versa. Then the new permutation p' is a permutation of the multiset consisting of k copies of 2 and $n-k$ copies of 1, so it is a permutation counted by $\begin{bmatrix}\mathbf{n}\\\mathbf{n-k}\end{bmatrix}$. Furthermore, $i(p) = i'(p)$, proving our claim.

30. The term $\begin{bmatrix}\mathbf{n-1}\\\mathbf{k}\end{bmatrix}$ enumerates permutations that end in a 2, and the term $q^{n-k}\begin{bmatrix}\mathbf{n-1}\\\mathbf{k-1}\end{bmatrix}$ enumerates permutations that end in a 1.

31. It is easy to prove that $p(m,n,q)$ satisfies the same recursive relation as $\begin{bmatrix}\mathbf{m+n}\\\mathbf{m}\end{bmatrix}$. Indeed, if the Ferrers shape of a partition fits into an $m \times n$ rectangle, then there are two possibilities: Either the partition has

at most $m-1$ parts, and then its Ferrers shape fits even into an $(m-1)\times n$ rectangle, or the partition has m parts, and then, after removing the first column of its Ferrers shape, its remaining shape fits into an $m\times(n-1)$ rectangle. These two cases correspond to the two summands in the recursive relation satisfied by the Gaussian coefficients.

32. We prove the statement by induction on n, the initial case of $n=1$ being trivial. We will say that there is one permutation of length 0 that avoids 132.

    Now assume that we know the statement for all nonnegative integers less than n. Suppose we have a 132-avoiding n-permutation in which the entry n is in the ith position. Then it is clear that any entry to the left of n must be larger than any entry to the right of n, otherwise the two entries violating this condition and the entry n would form a 132-pattern. Moreover, by our induction hypothesis, there are C_{i-1} possibilities for the substring of entries to the left of n, and C_{n-i} possibilities for that to the right of n. Summing over all allowed i, we get the following recursion:

$$C_n = \sum_{i=0}^{n-1} C_{i-1}C_{n-i}, \tag{4.21}$$

    and we know from (3.25) that this is the recursion of the Catalan numbers.

33. If the n-permutation p contains the pattern q, then the reverse of p will contain the reverse of q. This implies $S_n(132) = S_n(231)$. Similarly, if p contains the pattern q, then the complement of p will contain the complement of q. Therefore, $S_n(132) = S_n(213)$. Finally, $S_n(213) = S_n(312)$ by taking reverses. So all four patterns 132, 213, 231, 312 are avoided by C_n permutations of length n.

    This is not even the end of the story, since we also have $S_n(123) = S_n(321) = C_n$. The latter is somewhat harder to prove (Supplementary Exercise 39).

34. Let p_i, p_j, and p_k form a 312-pattern in p. If there are several 312-patterns in p, choose the one for which $j-i$ is minimal. In that case, we claim that $j-i=1$ must hold, meaning that p_i and p_j are consecutive entries of p. Indeed, assume p_t is located between p_i and p_j. Then p_t cannot be larger than p_k, since then $p_tp_jp_k$ would form a 312-pattern, contradicting the minimality of $j-i$. Similarly, p_t cannot be smaller than p_k, since then again $p_ip_tp_k$ would form a 312-pattern. So p_t simply cannot exist, meaning that p_i and p_j are consecutive.

35. We prove the statement by induction on n, the case of $n=1$ being

obvious. Now assume that the statement is true for $n-1$, and prove it for n. Say $n-1$ tourists are already sitting at tables, and wine has been served to those tables. Now the last tourist comes in. He can either sit at a new table, and get red wine or white wine, or sit at one of the tables with tourists, choosing his left neighbor in one of $n-1$ ways. Since the first scenario yields $2n!$ possible outcomes, and the second scenario yields $(n-1)n!$ possible outcomes, then altogether we have $(n+1)n! = (n+1)!$ possible outcomes, as claimed.

Note that we have essentially used the fact that the last tourist can take $n+1$ different courses of action.

36. (a) Since all permutations of the vertices define a symmetry, the regular tetrahedron has 24 symmetries.

    (b) There are 12 symmetries that can be obtained by a series of rotations. To see this, take an axis that is perpendicular to one of the faces and contains the fourth vertex. There are two rotations around that axis, by 120 and 240 degrees. There are four axes like this, leading to $4 \cdot 2 = 8$ symmetries. Following a rotation by a rotation around another axis gives us a symmetry that interchanges the elements of two pairs of vertices. There are three possible pairs of pairs, so there are three such symmetries. Finally, the identity is certainly a rotation. So there are $8+3+1 = 12$ symmetries that can be obtained by a series of rotations.

    (c) Part (b) shows that the symmetries that can be obtained by a series of rotations have cycle types $(4,0,0,0)$, $(1,0,1,0)$, or $(0,2,0,0)$, so they are even permutations. Since there are 12 of them, they are *all* even permutations of S_4. So a symmetry of a regular tetrahedron can be obtained by rotations if and only if it is an even permutation.

37. We claim that

$$H(x) = \prod_{i\geq 1} \left(\frac{1}{1-x^i} \right)^{1/i}.$$

Indeed, to get an object that we are enumerating, partition $[n]$ into blocks, index them by positive integers i, and then on block i, take a permutation whose cycles are *labeled* i, and have length divisible by i. The generating function for the number of ways to do this on block i is then $(1-x^i)^{-1/i}$, (see Exercise) for details, and the proof is immediate by the Product formula. This result and its generalizations appeared in [1].