(b) Prove that

$$Co_n = 2^{\binom{n}{2}} - \frac{1}{n}\sum_{k=1}^{n-1} k\binom{n}{k} Co_k \cdot 2^{\binom{n-k}{2}}.$$

30. Let $G$ be any graph on vertex set $[n]$, with loops and multiple edges allowed. Let $A(G)$ be the $n \times n$ matrix whose $(i,j)$-entry is equal to the number of edges between $i$ and $j$. Then $A(G)$ is called the *adjacency matrix* of $G$.

    Prove that the $(i,j)$-entry of $A(G)^k$ is equal to the number of *walks* of length $k$ from $i$ to $j$ for any positive integers $k$.

31. Let $S(x) = \sum_{n\geq 1} s_n x^n$, where $s_n$ is the number of rooted plane trees in which every nonleaf vertex has at least two children. Find a closed form for $S(x)$.

32. How many rooted plane trees are there on $n$ unlabeled vertices in which each vertex has at most two children? Use the Lagrange Inversion Formula in your solution.

## 5.10 Solutions to exercises

1. Let $G$ have $n$ vertices. If $G$ has an isolated vertex, then the maximum degree in $G$ is $n-2$, meaning that the possible degrees of vertices are $0, 1, \cdots, n-2$. This is $n-1$ possibilities, so our claim follows by the pigeonhole principle.

   If $G$ has no isolated vertices, then the possible degrees of vertices are $1, 2, \cdots, n-1$, which is again $n-1$ possibilities, and we conclude as above.

2. Let $W$ be a walk from $x$ to $y$. If $W$ is a path, we are done. Otherwise, there is at least one vertex that occurs at least twice in $W$. Let $z$ be the first such vertex. Then let us remove the part of $W$ that is between the first and the second occurrences of $z$ to get the new walk $W'$, which still connects $x$ and $y$. Now repeat the same procedure for $W'$ instead of $W$, and keep doing this. The procedure will eventually have to stop, since each step decreases the number of edges in $W$. When the procedure stops, there are no more repeated vertices, so we have a path from $x$ to $y$.

3. If $G$ is a tree, then it is connected, so there is at least one path from $x$ to $y$. If there were two, say $p$ and $p'$, then the symmetric difference of these two paths would contain a cycle. (In case you forgot, the symmetric difference of two sets is the set of elements

that belong to *exactly one* of the two sets.) Indeed, each component of this symmetric difference would be a graph in which each degree is two, and such a graph is a cycle.

Conversely, if $G$ has the mentioned property, then $G$ is connected and cycle free, since, if it had a cycle, there would be two paths between any two points of that cycle. Therefore, $G$ is a tree.

4. Prove this by assuming the opposite, that is, that there are two vertices, $x$ and $y$, that are closest to $v$ in $P$, say at distance $d$. Then there would be two paths from $v$ to $x$. One would be the shortest path of length $d$, and one would be the path of length $d+k$ that first goes from $v$ to $y$, then from $y$ to $x$ along $P$. (Why is that a path?) This contradicts the property of trees we proved in the previous exercise.

5. There are $10^8$ trees on 10 labeled vertices. Each of them can be isomorphic to at most $10! < 3.7 \cdot 10^6$ others (keep the tree fixed and permute the labels). Therefore, our claim follows by the pigeonhole principle.

6. This is similar to the proof of Theorem 5.24. We can again use the circular parking lot. The favorite spot of the first car can be any element of $[n+1]$, but there are only $n$ possibilities (all but the favorite spot of the preceding car) for the favorite spot of each subsequent car. Therefore, there are $(n+1)n^{n-1}$ possible sets of parking preferences, and again, $1/(n+1)$ of them lead to parking functions. Therefore, the answer is $n^{n-1}$.

7. By induction on $n$. For $n = 1$ the statement is trivially true. Assume that we know the statement for $n-1$, and prove it for $n$. Let $f$ be a parking function on $[n]$. Let $i$ be the car that will end up parking in spot $n$ if $f$ governs the parking procedure.

   Then, omitting car $i$, its parking preference, and the last spot, we get a parking function $f'$ on $[n-1]$, for which the induction hypothesis holds. That is, we know that the number of unsuccessful parking attempts for all cars excluding $i$ is $\binom{n}{2} - \sum_{j \neq i} f(j)$. If we have $f(i) = a$, then car $i$ had exactly $n-a$ unsuccessful parking attempts. Adding it to the previous expression, our claim is proved.

8. To solve this enumeration problem, we first characterize the possible ordered degree sequences. We claim that, for $n \geq 2$, a sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ is a possible sequence if and only if

   (a) $\sum_{i=1}^{n} d_i = 2n-2$, and
   (b) $d_{n-1} = d_n = 1$.

   On one hand, both conditions are necessary. Indeed, the first is equivalent to the graph having $n-1$ edges, and if the second one did not hold, we would have $d_i \geq 2$ for $i \leq n-1$, leading to $\sum_{i=1}^{n} d_i \geq 2n-1$, a contradiction since our tree has $n-1$ edges.

In order to prove that the two conditions together are sufficient, we use induction on $n$. If $n = 1$, then the only sequence allowed by the conditions is $d_1 = d_2 = 1$, and there is indeed a tree with that ordered degree sequence. Assume the statement is true for $n-1$, and prove it for $n$.

Let $S = d_1 \geq d_2 \geq \cdots \geq d_n$ be any allowed sequence with $n \geq 3$. Then there is a $d_i$ in the sequence that is larger than 1. Choose the last such $d_i$. Now remove $d_n$ from the sequence, and decrease $d_i$ by 1. Then the obtained sequence $S'$ satisfies the two criteria for $n-1$ (the sum of the elements is $2n-4$, and they are all positive, so the last two must be 1); therefore, the induction hypothesis applies. That is, there is a tree on $n-1$ vertices whose ordered degree sequence is $S'$. Adding a new vertex and connecting it to the vertex corresponding to $d_i$, we get a tree with ordered degree sequence $S$.

This completes the characterization of allowed sequences. For $n \geq 3$, their number is equal to the number of partitions $2n-4$ into $n-2$ parts (since the $(n-1)$st and $n$th parts must be 1).

9. (a) This is similar to the situation for parking functions. Roughly speaking, what causes a problem is when too many cars favor spots with high labels. The process will fail if more than one car wants the last spot, or in general, when more than $j$ cars want a spot among the $j$ last spots for any $j \in [n-k+1]$. (There are only $n-k+1$ cars, so we do not have to consider larger values of $j$.)

   We claim that, if, for any $j \in [n-k+1]$, there are at most $j$ cars who want to park in one of the last $j$ spots, then $f$ is a $k$-shortened parking function. The reader is invited to prove this claim on her own. (Reviewing the proof of Lemma 5.23 may be helpful.) Note that the difference lies in the weaker condition on $j$. Indeed, here we only need that the condition holds for all $j \in [n-k+1]$, not all $j \in [n]$. Also note that an alternative description of the sufficient and necessary condition for $f$ to be a $k$-shortened parking function is that the vector $(a_1, a_2, \cdots, a_{n-k+1})$ obtained by the nondecreasing rearrangement of the values $f(1), f(2), \cdots, f(n-k+1)$ is coordinate-wise smaller than or equal to the vector $(k, k+1, \cdots, n)$.

   (b) Consider again the circular parking lot of the proof of Theorem 5.24. We have $n+k-1$ cars, and they have parking preferences from $[n+1]$. Therefore, $k$ spots will remain empty. If one of the empty spots is spot $n+1$, then $f$ is a $k$-shortened parking function. We claim that that will happen in $k/(n+1)$

of all cases. Indeed, let $\mathbf{v} = (f(1), f(2), \cdots, f(n-k+1))$ be the vector of values of any function $f : [n-k+1] \to [n+1]$. For $i \in [n]$, define $\mathbf{v_i} = (f(1)+i, f(2)+i, \cdots, f(n-k+1)+i)$, where addition is meant modulo $n+1$. That is, the $\mathbf{v_i}$ are the circular translates of $\mathbf{v}$. Then exactly $k$ of the $n+1$ vectors $\mathbf{v_i}$ will not contain $n+1$ as a coordinate since there are exactly $k$ circular translations (rotations) that take $n+1$ into one of the $k$ spots that do not appear in $\mathbf{v}$. This proves our claim.

As the number of all functions $f : [n-k+1] \to [n+1]$ is certainly $(n+1)^{n-k+1}$, the number of $k$-shortened parking functions $f : [n-k+1] \to [n]$ is $k(n+1)^{n-k}$.

10. First choose the $k$ positions in which the $k$ values equal to 1 will be. This can be done in $\binom{n}{k}$ ways. For the remaining $n-k$ positions, we need values so that the vector obtained by their nondecreasing rearrangement $(b_1, b_2, \cdots, b_{n-k})$ consists of coordinates larger than 1 and is coordinate-wise smaller than $(k+1, k+2, \cdots, n)$. Equivalently, the vector $(b_1 - 1, b_2 - 1, \cdots, b_{n-k} - 1)$ must consist of coordinates that are positive integers and must be coordinate-wise smaller than $(k, k+1, \cdots, n-1)$. In other words, it must be the nondecreasing rearrangement of a $k$-shortened parking function $f : [n-k] \to [n-1]$. We know from part (b) of the previous exercise that this is possible in $kn^{n-1-k}$ ways. Therefore, we have

$$P(n,k) = \binom{n}{k} kn^{n-1-k} = \binom{n-1}{k-1} n^{n-k}.$$

11. Let $p$ be a parking function on $[n]$. Find the smallest $a_1$ so that $p$ contains only $a_1$ values that are elements of $[a_1]$, that is, the smallest $a_1$ for which the conditions of $p$ being a parking function are satisfied in a tight way. If there is no such $a_1$, then $p$ is prime, and the statement holds, with $k = 1$. Otherwise, there are $a_1$ cars whose parking preferences are within $[a_1]$, so their parking preferences form a parking function on $[a_1]$. Let $s_1$ be the set of these cars, and let $p_1$ be the parking function they determine. Then $p_1$ is prime, since $a_1$ was the *smallest* integer for which the parking function criteria held tightly, so a 1 can be omitted from $p_1$.

    Then iterate this procedure. That is, subtract $a_1$ from all remaining values of $p$ to get a parking function $p'$ on $[n - a_1]$. Look again for the smallest $a_2$ for which there are only $a_2$ values of $p'$ that are elements of $[a_2]$, and so on. The procedure will stop when we get to a *prime* parking function; that function will be $p_k$.

    For instance, if $p = (1, 7, 1, 6, 2, 1, 4, 7, 7)$, then we have $a_1 = 5$. The values of $p$ that are from $[5]$ are in the first, third, fifth, sixth, and seventh position, so $s_1 = \{1, 3, 5, 6, 7\}$ and $p_1 = (1, 1, 2, 1, 4)$. This leaves us with $p' = (2,1,2,2)$, which yields $a_2 = 1$. So $p_2 = (1)$ and $s_2 = \{4\}$, since the second value of $p'$ was the fourth value of $p$. This in turn leaves us with $p'' = (1,1,1)$, which is a prime parking function. So the decomposition we obtain is

$$(1,1,2,1,4),(1),(1,1,1),\{1,3,5,6,7\},\{4\},\{2,8,9\}.$$

In order to recover the original parking function $p$ from its decomposition $(p_1, p_2, \cdots, p_k, s_1, s_2, \cdots, s_k)$, simply put the values of $p_i$ into the positions specified by $s_i$, then add $|s_1| + |s_2| + \cdots + |s_{i-1}|$ to each value of $p_i$.

12. Take a function counted by $P(n,k)$ and insert a 1 after any of its $n$ values, or at the beginning. What we get is a prime parking function on $[n+1]$, containing $k+1$ values equal to one, and therefore a function counted by $PP(n+1,k+1)$. Each such function will be obtained $k+1$ times, since each of its entries equal to 1 can be the recently inserted 1. This shows

$$PP(n+1,k+1) = \frac{n+1}{k+1}P(n,k).$$

13. A prime parking function on $[n+1]$ must have at least two and at most $n+1$ values equal to 1. Using the result of the previous exercise and that of Exercise 10, we get that

$$\begin{aligned} PP(n+1) &= \sum_{k=1}^{n} PP(n+1,k+1) \\ &= \sum_{k=1}^{n} \frac{n+1}{k+1} P(n,k) \\ &= \sum_{k=1}^{n} \frac{n+1}{k+1} \cdot \binom{n-1}{k-1} n^{n-k} \\ &= \sum_{k=1}^{n} k \cdot \binom{n+1}{k+1} n^{n-k-1}. \end{aligned}$$

Dividing both sides by $n^{n-1}$, we get the identity

$$\frac{PP(n+1)}{n^{n-1}} = \sum_{k=1}^{n} \frac{k}{n^k} \cdot \binom{n+1}{k+1}.$$

Exercise 28 of Chapter 1 shows that the right-hand side is equal to $n$, and our result is proved.

14. We have seen that each parking function is decomposable into an *ordered* partition of prime parking functions. Therefore, the compositional formula (Theorem 3.36) applies, with $B(x) = \sum_{k \geq 0} k! \frac{x^k}{k!} = 1/(1-x)$.

15. For $n \leq 3$, there is only one unlabeled tree on $n$ vertices, namely, the path on $n$ vertices. For $n = 4$, we either have a vertex with degree three, and then we get a star, or we do not, and then we have a path. So $u(4) = 2$. If $n = 5$, then the maximum degree may be four (and then we get the star), or three (and then we get a star with an edge added to one of the leaves), or two (and then we get a path). So $u(5) = 3$. If $n = 6$, then the same holds for the cases when the maximum degree is five, or four, or two. However, when the maximum degree is three, there are several possibilities. Either there are two vertices with degree three, and then we get the tree shown on the left of Figure 5.48, or there is just one such vertex, leading to the trees on the right and at the bottom of Figure 5.48 (in one, the new vertices not connected to the vertex $A$ of degree three are both at distance two from $A$; in the other one, one of them is at distance two from $A$, and the other one is at distance three). Therefore, $u(6) = 6$.

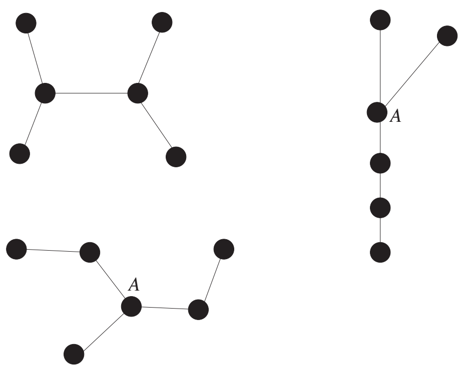


**Figure 5.48**
The three unlabeled trees on six vertices and maximum degree three.

16. We ask the reader to verify that the vertex at the center of the graph $G$ has to be mapped into itself by every automorphism (which is the only vertex of degree six, and isomorphisms preserve degrees). Then any permutation of the remaining vertices is an automorphism. Therefore, $G$ has $6! = 720$ automorphisms. In $H$, the two vertices of degree three can be interchanged, and the two vertices of degree two can be interchanged, so $|Aut(H)| = 4$. Finally, in $J$, let $A$ and $B$ be two neighboring vertices. Then $f(A)$ and $f(B)$ must be neighbors for any automorphisms $f$. We have five choices for the image of $A$, after which we have two choices for the image of $B$, and then our hands are tied. Therefore, $J$ has 10 automorphisms.

17. By Theorem 5.7, this is equivalent to finding a graph on six vertices

that has only one automorphism, the trivial one. Figure 5.49 shows such a graph.

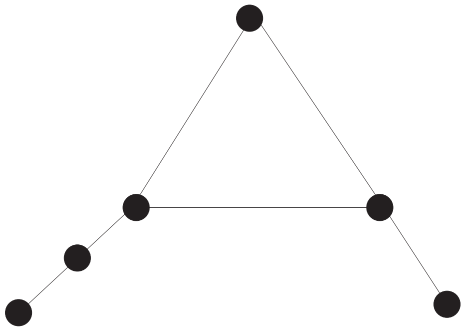

**Figure 5.49**
This graph has no nontrivial automorphisms.

18. We construct a bijection $f$ from $S_n$ into the set $M_n$ of decreasing trees on $n+1$ vertices. Let $p \in S_n$ and let $p = p_1 p_2 \cdots p_n$. Each entry $p_i$ will correspond to a vertex of $f(p)$. The unique parent of the vertex corresponding to $p_i$ will be the vertex corresponding to $p_j$, where $p_j$ is the *rightmost* entry on the left of $p_i$ so that $p_j > p_i$. If there is no such entry $p_j$, then the parent of $p_i$ is the extra vertex $n+1$. For instance, on the left of Figure 5.47, we see $f(34251)$, and on the right of that figure, we see $f(23514)$.

    To start with, $f(p) \in M_n$ since our definition results in a decreasing labeling of the obtained tree. Now we prove that $f$ is a bijection by showing that it has an inverse. Let $M \in M_n$. Then the definition of $f$ implies that the children of the root vertex $n+1$ must be the left-to-right maxima of $p$ since they are precisely the entries that are larger than everything on their right, so it is for these entries that we will not find a $p_j$ for a parent. So, if $f(p) = M$, then we can read off the set of left-to-right maxima of $p$ from $M$. Because the left-to-right maxima of a permutation form an increasing sequence, we know their order as well. Say the left-to-right maxima are $m_1 < m_2 < \cdots < m_k$. Then the subtree of $M$ that has $m_i$ for its root uniquely describes the string of $p$ starting in $m_i$ and ending right before $m_{i+1}$, by iterated applications of this argument.

19. We have seen in the previous exercise that there is a bijection from the set of $n$-permutations with $k$ left-to-right maxima and onto the set of our trees. Since the number of the first one is $c(n,k)$ by the transition lemma, this is the number of the latter as well.

20. (a) A 132-avoiding permutation $p$ can be cut into blocks so that each entry of each given block is larger than all entries on the

right of that block. For instance, 78453612 can be cut into blocks as 78|4536|12. In general, the first cut comes immediately on the right of the maximal entry, then we proceed recursively to the right. If $p$ ends with its maximal entry, then $p$ is just one block.

An induction proof along the lines of the proof of Theorem 5.28 is now straightforward, with permutations ending in $n$ corresponding to trees in which the root has only one child.

(b) The bijection of part (a) will work. This can be seen by induction, since the number of leaves of a rooted plane tree is just the sum of the numbers of leaves in the subtrees of the root. Similarly, the number of left-to-right minima of a 132-avoiding permutation is just the sum of the numbers of left-to-right minima in the blocks.

21. The definition of 132-avoiding permutations says that $p$ is 132-avoiding if we cannot find three entries $a$, $b$, and $c$ in $p$ so that $a$ is the leftmost of them, $c$ is the rightmost of them, and $b$ is the largest of them, while $a < c$.

This means that, in the binary plane tree $T(p)$ of $p$, all entries of $T(L)$ must be larger than all entries of $T(R)$. (Otherwise, if entries $a$ and $c$ violate this constraint, then $anc$ is a 132-pattern.) Recursively, the same must hold for each node of $T(p)$. That is, for each node of $T(p)$, the labels of any vertex in the left subtree of that node must be larger than the label of any vertex in the right subtree of that node.

In other words, once the unlabeled tree $T(p)$ is given and we know that $p$ is 132-avoiding, we can recover $p$ by selecting the labels so that the conditions of the previous paragraph hold.

For a more explicit description of this argument, say that we have already defined the bijection $f$ for binary plane trees on fewer than $n$ vertices. Let $T$ be a binary plane tree on $n$ vertices, with left subtree $T_L$ and right subtree $T_R$. Then we define $f(T) = f(T_L)'nf(T_R)$, where $f(T_L)'$ is obtained by adding $|T_R|$ to each entry of $f(T_L)$.

22. We claim that, for any $n$-permutation $p$, the number of descents of $p$ is equal to the right edges of $T(p)$. This claim is easy to prove by induction (look at the two subtrees of $T(p)$). Then the claim of the exercise is straightforward, since $T(p)$ can be reflected through a vertical axis.

23. The statement will be proved if we can prove that there are as many 132-avoiding permutations with $k-1$ descents as there are such permutations with $k$ left-to-right minima.

We claim that even more is true, that is, these two sets of permutations are not simply *equinumerous*, they are *identical*. In fact, we

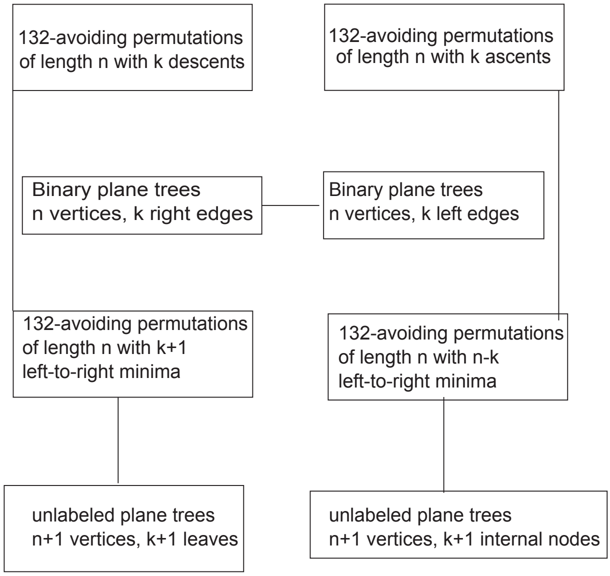


**Figure 5.50**
The bijections proved in Exercises 20–23.

claim that if $p = p_1p_2 \cdots p_n$ is 132-avoiding, then $p_i$ is a left-to-right minimum of $p$ if and only if either $i = 1$ or $i-1$ is a descent of $p$. In other words, the left-to-right minima are precisely the ends of the descents (and of course the leftmost entry).

The "only if" part is trivial. To see the "if" part, assume the contrary, that is, that there exists $j < i$ so that $p_j < p_i$. Then $p_jp_{i-1}p_i$ is a 132-pattern, which is a contradiction.

Figure 5.50 shows all the bijections that were proved in the last several exercises. We point out again that the number of elements of each of these sets is the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. See the remark after Theorem 5.29 for references about these numbers.

24. These functions are in bijection with rooted forests on $[n]$ having $k$ components. Therefore, by Theorem 5.19, their number is $a_{n,k} = \binom{n}{k}kn^{n-1-k}$.

25. It follows from the result of the previous exercise that we have $a_{n,k} = \binom{n-1}{k-1}n^{n-k}$. Therefore, by the binomial theorem,

$$\begin{aligned} A_n(x) &= \sum_{k=1}^{n}\binom{n-1}{k-1}n^{n-k}x^k = x\sum_{i=0}^{n-1}\binom{n-1}{i}n^{n-1-i}x^i \\ &= x(x+n)^{n-1}. \end{aligned}$$

26. If $g$ is a proper coloring, then it defines a unique orientation $A$ of $G$ that satisfies the second criterion (edges point toward the smaller color), and that orientation is automatically acyclic.

    Conversely, if $(g, A)$ satisfies the criteria, then $g$ is a proper coloring since no edge has monochromatic vertices. For each $g$, there is one $A$ so that $(g, A)$ satisfies the criteria. This bijectively proves our claim.

27. First, $n$ and $m$ are nonnegative, since Theorem 5.46 implies that $\chi_G(n) \neq 0$ for $n < 0$. Indeed, $\bar{\chi}_G(n) \geq 1$ for $n > 0$, because, when we are looking for colorings counted by $\bar{\chi}_G(n) \neq 0$, we can always color all vertices with the same color.

    Second, if $\chi_G(m) = 0$, then there is no proper coloring of $G$ using only colors from $[m]$, but then, of course, there is no proper coloring of $G$ using only colors from $[i]$ where $i \leq m$.

28. We claim that the number of edges of the simple graph $G$ on vertex set $[m]$ is equal to $-1$ times the coefficient of $n^{m-1}$ (the term following the leading term) in $\chi_G$. We will prove this statement by induction on the number of edges, using Proposition 5.40.

    If $G$ is the empty graph, then $\chi_G(n) = n^m$, and the claim is true. Now let us assume that the claim is true for graphs with $k-1$ edges, and let $G$ have $k$ edges. Then Proposition 5.40 says that $\chi_G(n) = \chi_{G_e}(n) - \chi_{G^e}(n)$. The term of degree $m-1$ is the leading term in $\chi_{G^e}(n)$, and as such, its coefficient is 1 (Supplementary Exercise 26), and the term of degree $m-1$ is the second term in $\chi_{G_e}(n)$, so, by our induction hypothesis, its coefficient is $k-1$. So the coefficient of $n^{m-1}$ on the left-hand side is $(k-1)+1 = k$, as claimed.

29. (a) The summand indexed by $k$ counts the rooted graphs on $[n]$ in which the root is in a component of size $k$.

    (b) The summand indexed by $k$ on the right-hand side is the number of rooted graphs on $[n]$ in which the root is in a component of size $k$. If $k < n$, then the graph is not connected. Dividing the sum by $n$ will count *unrooted* graphs that are not connected, and then the statement is proved by the subtraction principle.

30. Use induction on $k$ and the definition of matrix multiplication.

31. Just as we proceeded in Section 5.5.2.1, remove the root of such a tree, and notice that you either get the empty tree or a sequence of at least two trees. Therefore, we have

$$S(x) = x\left(1 + \sum_{n\geq 2} S(x)^n\right) = x + \frac{S(x)^2}{1 - S(x)}.$$