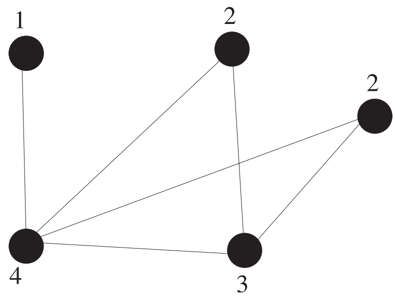

Figure 5.5
Degrees in a graph.

5.1 Trees and forests

In this section, we will count trees and other graphs whose vertex set is the set $[n]$.

5.1.1 Trees

While being connected is certainly a very desirable property when building any sort of network, it has its own costs. Therefore, it is a natural direction of research to study graphs that are connected but have no "redundancy." To make that notion more precise, we say that a graph G is *minimally connected* if G is connected, but if we remove any edge of G, then G will no longer be connected. As we have mentioned, graph B in Figure 5.1 is minimally connected.

Minimally connected graphs play a central role in graph theory. In order to understand their importance, we prove the following lemma, which provides three equivalent definitions for the same notion. Note how simple and natural all three of them are. We define a *cycle* as a walk whose starting point is the same as its endpoint, but which otherwise has no repeated vertices. For instance, C_1 and C_2 in Figure 5.1 are both cycles.

Lemma 5.3 *Let G be a connected simple graph on n vertices. Then the following are equivalent.*

(i) The graph G is minimally connected.

(ii) There are no cycles in G.

(iii) The graph G has exactly $n-1$ edges.

Proof:

(i)⇒ (ii) Assume there is a cycle C in G. Then G cannot be minimally connected since any one edge e of C can be omitted, and the obtained graph G' is still connected. Indeed, if a path uv used the edge e, then there would be a walk from u to v in which the edge e is replaced by the set edges of C that are different from e.

(ii)⇒ (iii) Pick any vertex x in G and start walking in some direction, never revisiting a vertex. As there is no cycle in G, eventually we will get stuck, meaning that we will hit a vertex of degree 1. This means that a connected simple graph with no cycles contains a vertex of degree 1. Removing such a vertex (and the only edge adjacent to it) from G, we get a graph $G*$ with one less vertex and one less edge, and the statement is proved by induction on n.

(iii) ⇒ (i) This statement says that, if a simple graph on n vertices is connected and has $n-1$ edges, then it is minimally connected, that is, a graph on n vertices and $n-2$ edges cannot be connected. Let us assume that this is not true, and let H be a counterexample with a minimum number of vertices. The reader is invited to check that H must have more than three vertices for this. As H has $n-2$ edges, there has to be a vertex y of degree 1 in H, otherwise H would need to have at least n edges (since the sum of the degrees of its vertices would be at least $2n$). Removing y from H, we get an even smaller counterexample for our statement, which is a contradiction.

◊

Graphs that satisfy the equivalent conditions mentioned in Lemma 5.3 are of central importance in graph theory. Therefore, they have their own name.

Definition 5.4 *A simple graph that is minimally connected is called a* tree*.*

Equivalently, a simple graph that is connected and cycle-free is called a tree. Equivalently, a simple graph of n vertices that is connected and has $n-1$ edges is called a tree.

A graph whose connected components are all trees is called, not surprisingly, a *forest*. Trees can look quite different from each other. See Figure 5.6 for some examples.

In the proof of Lemma 5.3, we spoke about vertices of degree 1 in trees. Such vertices will be called *leaves*. So our argument in the second part of that proof showed that each tree has at least one leaf. This statement can be significantly refined. See Supplementary Exercises 4 and 5 for some questions in that direction.