both for rooted forests on $[n+1]$ and for parking functions on $[n+1]$. The key word here is *combinatorially*, since the above identity could be proved purely by algebraic manipulations, but that would not give us the insight we are looking for.

First, let F be a rooted forest on $[n+1]$. Let F_1 be the component of F that contains the vertex 1. Say F_1 has $i+1$ vertices. Then we have $\binom{n}{i}$ ways to choose the vertices other than 1 for F_1, and we have $P(i)$ ways to choose a tree on these $i+1$ vertices. Once that is done, any of these $i+1$ vertices can be the root of F_1. Finally, we have $P(n-i)$ ways to choose a rooted forest on the remaining $n-i$ vertices. Therefore, by the product principle, we indeed have $\binom{n}{i}(i+1)P(i)P(n-i)$ rooted forests on $[n+1]$ in which the vertex 1 is in a component of size $i+1$. Summing over all allowed i, we get (5.5).

Second, let f be a parking function on $[n+1]$. Say that when the last car arrives, spot $i+1$ is free for some $i \in [0,n]$. That means that there is a set S of i cars who parked in the first i spots, so their parking preferences formed a parking function on $[i]$. If we subtract $i+1$ from the parking preferences of the remaining $n-i$ cars, then they form a parking function on $[n-i]$. (Note that none of these $n-i$ cars could have had a parking preference in $[i+1]$ since the spot $i+1$ would not be free.) Finally, f is a parking function, so $f(n+1) \leq i+1$. That is, we have $i+1$ possibilities for $f(n+1)$. Since we have $\binom{n}{i}$ possibilities for S, formula (5.5) is proved again by the product principle and by summing over all allowed i.

Quick Check

1. How many parking functions $f : [n] \rightarrow [n]$ are there that satisfy $\sum_{i=1}^{n} f(i) = \binom{n+1}{2}$?
2. How many parking functions $f : [n] \rightarrow [n]$ are there that satisfy $\sum_{i=1}^{n} f(i) = \binom{n+1}{2} - 1$?
3. What is the number of functions $f : [n] \rightarrow [n]$ such that the short diagram of f contains exactly one cycle, and that cycle is of length k?

5.3 When the vertices are not freely labeled

There are some situations when we restrict the labels of the vertices of trees in some natural way. In this section, we consider some of the ensuing enumeration problems.