

The reader is probably wondering what the big deal is, since reflecting a tree through an axis is not all that interesting. What is interesting, however, is the fact that this symmetry can be translated into all 150 interpretations of Catalan numbers, and in many of those interpretations, it will prove symmetries that are far less obvious than the one proved in Proposition 5.34.

For instance, in Exercises 21 and 22 we show how to use Proposition 5.34 to prove the much less obvious fact that the number of 132-avoiding n -permutations with k descents is the same as the number of 132-avoiding n -permutations with k ascents. In case you do not remember, 132-avoiding permutations were defined in Exercise 32.

Then, Exercises 18 and 20 connect the topic of the previous subsection, rooted plane trees, to these objects as well. Therefore, the reader solving all these exercises can conclude in Exercise 23 that there are as many rooted plane trees on $n + 1$ vertices with k internal nodes as there are with k leaves. This is what we claimed in Theorem 5.29.

In general, a similar symmetry can be found for each interpretation of the Catalan numbers. Furthermore, for each interpretation of the Catalan numbers, we can find a natural partition of the set of objects counted by the Catalan numbers so that the sizes of the blocks of the partition are given by the Narayana numbers, defined by (5.6). The interested reader should consult [72].

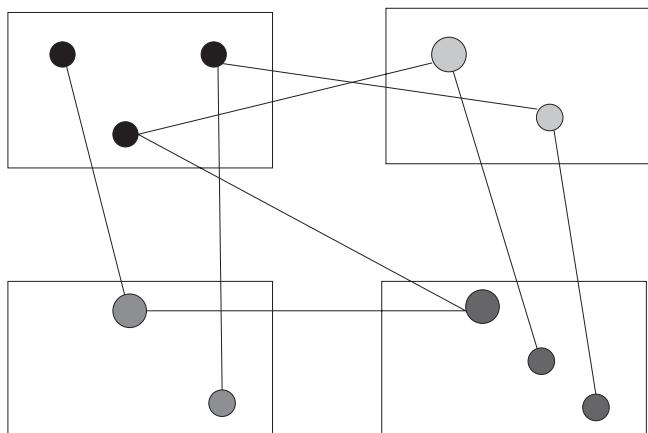
Quick Check

1. What is the number of rooted plane trees on $n + 1$ unlabeled vertices in which no nonroot vertex has degree higher than 1?
2. What is the number of decreasing binary trees on vertex set $[n]$ in which no vertex has two children?
3. What is the number of binary plane trees on n vertices that have exactly two leaves?

5.4 Graphs on colored vertices

Let us assume that representatives of four countries convene for a series of negotiations. The negotiations will consist of a round of one-on-one conversations, each occurring between two people coming from different countries.

If we want to represent these negotiations by a graph, then the vertices will correspond to the negotiators, and there will be an edge between two vertices if the two corresponding people talked to each other. However, the graph we get will have a special structure. That is, there will be no edges between vertices corresponding to people from the same country. See [Figure 5.28](#) for an example.

**Figure 5.28**

Diplomats from the same country will not negotiate with each other.

This example shows that sometimes we want to study graphs in which we do *not* want edges between certain subsets of vertices. Another example of this is communication towers and the frequencies on which they broadcast. We can represent communication towers by vertices of a graph and join two vertices by an edge if the corresponding towers are closer to each other than a specified distance d . Towers that are closer to each other than d should not broadcast on the same frequency. Therefore, if we represent frequencies by colors, then the distribution of frequencies to the towers is equivalent to coloring the vertices of the graph so that adjacent vertices are of different colors.

The idea of colorings is an easy way to visualize such requirements, that is, to keep track of pairs of vertices that cannot be adjacent. In the example of the negotiators, we would color vertices corresponding to representatives of the same country with the same color (or we would color vertices corresponding to communication towers built close to each other with the same color) and then we would not connect vertices of the same color by edges.

Definition 5.35 *Let G be a graph. We say that a coloring of the vertices of G is proper if adjacent vertices are of different colors.*

There are two types of questions we are going to look at in this section:

1. Given a *graph* G and a number n of colors, how many proper colorings does G have using only those n colors?
2. Given a *colored vertex set*, how many graphs are there on that vertex set so that the coloring is proper?