be in a preimage of T and defines the values of f and g for the elements of C. For the elements of N, the value of $f(x)$ or $g(x)$ (whichever is meaningful) is the neighbor of x toward the Start-End path on the unique path from x to the Start-End path. ♢

Theorem 5.49 can be generalized to an arbitrary number of colors, as the following theorem shows.

Theorem 5.54 *Let $n_1+n_2+\cdots+n_k = n$, where the n_i are positive integers. Then the number of properly k-colored trees on n vertices, n_i of which are of color i and are bijectively labeled by the elements of $[n_i]$ (for all i), is*

$$n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}.$$

Note that this formula implies not only Theorem 5.49, but also Theorem 5.10, since we can choose $n_i = 1$ for all i. See [53] for a proof.

Quick Check

1. Find the chromatic polynomial of a path of n vertices.
2. Is it true that all trees on n vertices have the same chromatic polynomial?
3. What is the number of acyclic orientations of the complete graph K_n?

5.5 Graphs and generating functions

It is high time that we applied the powerful counting techniques that we learned in the previous chapter to enumerate various kinds of trees. We start by exploring a classic family of trees.

5.5.1 Trees counted by Cayley's formula

Rooted trees on $[n]$ have a nice recursive structure. If we cut off the root R of such a tree, we get a rooted forest, that is, trees that are rooted at the vertex which was a neighbor of R. In other words, rooted forests are built up from rooted trees. This structure can be expressed by a simple functional equation. We know from Cayley's formula that the number of rooted trees on