The connection between even graphs and Eulerian graphs is the same as the connection between all simple graphs and connected simple graphs. That is, in both of these pairs, the *connected components* of the first class of graphs are precisely the graphs from the second class. Let us express this fact by generating functions.

Set $b_n = 2^{\binom{n-1}{2}}$ for $n \geq 1$, and $b_0 = 1$. Then the exponential generating function of the numbers of even graphs on $[n]$ is $B(x) = 1 + \sum_{n\geq 1} 2^{\binom{n-1}{2}} \frac{x^n}{n!}$. Let $Eu(x) = \sum_{n\geq 1} Eu(n)\frac{x^n}{n!}$, where $Eu(n)$ denotes the number of Eulerian graphs on vertex set $[n]$. Then the exponential formula yields

$$B(x) = e^{Eu(x)},$$

$$Eu(x) = \ln(B(x)).$$

We can now use a software package to compute the first few coefficients of $Eu(x)$ just as we did for the generating function of connected graphs. We get that

$$Eu(x) = x + \frac{1}{3!}x^3 + \frac{3}{4!}x^4 + \frac{38}{5!}x^5 + \frac{720}{6!} + \cdots .$$

So there are three Eulerian graphs on vertex set [4], there are 38 Eulerian graphs on vertex set [5], and 720 Eulerian graphs on vertex set [6].

### Quick Check

1. Find the exponential generating function for the number of simple graphs on vertex set $[n]$ in which every vertex has degree 2.
2. Find the exponential generating function for the number of simple graphs on vertex set $[n]$ in which every vertex has degree at most 2.
3. Let $a_n$ be the number of rooted plane trees on $2n + 1$ unlabeled vertices in which every nonleaf vertex has two children. Find the ordinary generating function of the numbers $a_n$. Where have you seen that generating function before?

## 5.6 The Lagrange Inversion Formula

We have seen examples of trees varieties where *each* vertex of a tree is labeled, and of tree varieties where *no* vertex is labeled. Now we will consider trees where *only leaves* are labeled. A *k-phylogenetic tree* is a rooted non-plane trees whose *leaves* are bijectively labeled with the elements of the set $[n] = \{1, 2, \cdots, n\}$, and in which each non-leaf vertex has exactly $k$ children. See Figure 5.44 for the set of all three 2-phylogenetic trees on label set [3].

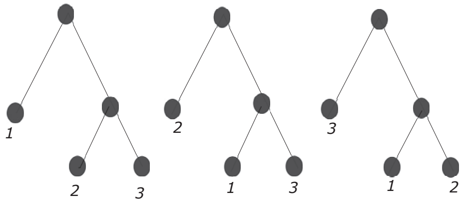

**Figure 5.44**
The three 2-phylogenetic trees on leaf set [3].

Let $t_{k,n}$ be the number of $k$-phylogenetic trees on leaf set $[n]$, and set $t_{k,0} = 0$. Let $T_k(x) = \sum_{n\geq 0} t_{k,n}\frac{x^n}{n!}$ be the exponential generating function of the sequence of these numbers.

Removing the root of such a tree, we get either the empty set, or an unordered set of $k$ such trees, leading to the functional equation

$$T_k(x) = x + \frac{T_k^k(x)}{k!}. \tag{5.16}$$

The form

$$T_k(x) - \frac{T_k^k(x)}{k!} = x \tag{5.17}$$

will be particularly useful for our purposes.

If $k = 2$, then (5.16) is the quadratic equation $T_2(x) = x + \frac{T_2^2(x)}{2}$. Solving this equation, we get the formula

$$T_2(x) = 1 - \sqrt{1-2x},$$

which implies that

$$t_{2,n} = (2n-3)!!.$$

However, for $k > 2$, we need stronger techniques to find an explicit formula for $t_n$.

To that end, we first define the notion of the *compositional inverse* of a power series.

**Definition 5.58** *Let $F(x)$ be a formal power series. We say that the formal power series $G(x)$ is the* compositional inverse *of $F(x)$ if $F(G(x)) = G(F(x)) = x$. In this case, we write $F^{\langle -1\rangle}(x) = G(x)$.*

**Example 5.59** *If $F(x) = 3x + 2$, then $F^{\langle -1\rangle}(x) = (x-2)/3$.*

**Solution:** Indeed, $F\left(\frac{x-2}{3}\right) = (x-2)+2 = x$. Also, $F^{\langle -1\rangle}(3x+2) = 3x/3 = x$.
$\diamond$

Note that for the compositions $F(G(x))$ and $G(F(x))$ of *formal power series* to be defined, it suffices to assume that the constant term of $F$ and the constant term of $G$ are 0. In what follows, we will work with formal power series that satisfy that condition. We point out that if $F$ has infinitely many nonzero coefficients and $G$ has a nonzero constant term, then $F(G(x))$ is not defined.

**Example 5.60** *If* $F(x) = \sum_{n\geq 1} x^n = x/(1-x)$, *then* $F^{\langle -1\rangle}(x) = \sum_{n\geq 0}(-1)^{n-1}x^n = x/(1+x)$.

**Solution:** We have

$$F\left(\frac{x}{1+x}\right) = \frac{(x/(1+x)}{1-[x/(1+x)]} = \frac{x}{1+x-x} = x. \tag{5.18}$$

◊

**Example 5.61** *If* $F(x) = \ln(1+x)$, *then* $F^{\langle -1\rangle}(x) = e^x - 1$.

The following lemma shows that for power series with 0 as a constant term, it is easy to decide if their inverse exists.

**Lemma 5.62** *Let* $F(x) = f_1x + f_2x^2 + \cdots$ *be a formal power series. Then* $F^{\langle -1\rangle}(x)$ *exists if and only if* $f_1 \neq 0$. *In that case,* $F^{\langle -1\rangle}(x)$ *is unique.*

**Proof:** If $f_1 = 0$, then $G(F(x))$ will have coefficient 0 for its linear term, so $G(F(x)) \neq x$. Now let us assume that $f_1 \neq 0$. Let us see if we can find a power series $G(x) = \sum_{n\geq 1} g_nx^n$ so that $G(F(x)) = x$. (The reader is invited to verify that such a power series $G(x)$ must have a 0 constant term.) Equating the coefficients of $x^n$ for $n = 1, 2, 3\cdots$ in $G(F(x))$ and $x$, we get

$$G(F(x)) = g_1F(x) + g_2F^2(x) + g_3F^3(x) + \cdots,$$

which implies that $G(F(x)) = x$ if and only if

$$\begin{aligned} g_1f_1 &= 1, \\ g_1f_2 + g_2f_1^2 &= 0, \\ g_1f_3 + g_2f_1f_2 + g_3f_1^3 &= 0, \end{aligned}$$

and so on. We can solve the first equation for $g_1$ as $f_1 \neq 0$. After that, we can solve the second one for $g_2$, the third one for $g_3$, and so on. There will never be a problem of dividing by 0, since in the $n$th equation in the array above, the coefficient of $g_n$ will always be $f_1^n \neq 0$. ◊

A *formal Laurent series* is a power series with finitely many negative exponents. That is, a formal Laurent series with complex coefficients is of the form $\sum_{i\geq m} a_ix^i$, where $m \in \mathbf{Z}$. We are now in a position to state and prove the main result of this section.

**Theorem 5.63** *Let $n$ and $\ell$ be positive integers, and let $F^{\langle -1\rangle}(x)$ be the compositional inverse of the power series $F(x)=\sum_{i\geq 1} f_i x^i$. Then*

$$n[x^n](F^{\langle -1\rangle}(x))^{\ell} = \ell[x^{n-\ell}]\left(\frac{x}{F(x)}\right)^{n}. \tag{5.19}$$

Note that in particular, the case of $\ell = 1$ gives us the coefficients of $F^{\langle -1\rangle}(x)$. This is the most frequently used special case of the Lagrange Inversion Formula.

The following proof is due to Joseph-Louis Lagrange himself. The interested reader can consult Chapter 5 of [72] for additional proofs.

**Proof:** Let

$$F^{\langle -1\rangle}(x)^{\ell} = \sum_{i\geq \ell} h_i x^i.$$

Substituting $F(x)$ for $x$ on both sides of the above equality, we get

$$x^{\ell} = \sum_{i\geq \ell} h_i F(x)^i.$$

Note that the last displayed formula contains the coefficients of $F^{\langle -1\rangle}(x)^{\ell}$, but not $F^{\langle -1\rangle}(x)$ itself. This is a significant step toward expressing these coefficients in terms of $F(x)$.

Let us take the derivative of both sides to get

$$\ell x^{\ell-1} = \sum_{i\geq \ell} i h_i F(x)^{i-1} F'(x).$$

After rearranging, this yields

$$\frac{\ell x^{\ell-1}}{F(x)^n} = \sum_{i\geq \ell} i h_i F(x)^{i-1-n} F'(x). \tag{5.20}$$

Let us compare the coefficients of $x^{-1}$ on both sides of (5.20). Note that if $i\neq n$, then

$$F(x)^{i-1-n}F'(x) = \frac{\left(F(x)^{i-n}\right)'}{i-n}.$$

In other words, if $i\neq n$, then $h_i F(x)^{i-1-n}F'(x)$ is a constant multiple of the *derivative of a Laurent series*, and as such, its coefficient for the $x^{-1}$ term is 0. (Since the antiderivative of $x^{-1}$ is not a power function.) Therefore, the only summand on the right-hand side of (5.20) that will contribute to the coefficient of $x^{-1}$ is the summand indexed by $i=n$. That coefficient is

$$\begin{aligned}
[x^{-1}]nh_nF(x)^{-1}F'(x) &= [x^{-1}]nh_n\left(\frac{h_1+2h_2x+3h_3x_+^2\cdots}{h_1x+h_2x_2+\cdots}\right)\\
&= [x^{-1}]nh_n\left(\frac{1}{x}+\cdots\right)\\
&= nh_n,
\end{aligned}$$

which is the expression on the left-hand side of (5.19), that was to be proved. Now returning to (5.20), notice that

$$[x^{-1}]\frac{\ell x^{\ell-1}}{F(x)^n} = [x^{n-\ell}]\frac{\ell x^{n\ell}}{F(x)^n},$$

which is the right-hand side of (5.19).

◊

Now we can return to our $k$-phylogenetic trees. Formula (5.17) shows that $F(T(x)) = x$, so $T_k(x) = F^{\langle -1\rangle}(x)$, where

$$F(x) = \left(x - \frac{x^k}{k!}\right)^{\langle -1\rangle}.$$

Therefore, we can compute the coefficients of $T_k(x)$ using Theorem 5.63.

That theorem yields

$$n[x^n]T_k^\ell(x) = \ell[x^{n-\ell}]\left(\frac{x}{x - \frac{x^k}{k!}}\right)^n.$$

From this, we compute

$$\begin{aligned}
[x^n]T_k^\ell(x) &= \frac{\ell}{n}[x^{n-\ell}]\left(1 - \frac{x^{k-1}}{k!}\right)^{-n} \\
&= \frac{\ell}{n}[x^{n-\ell}]\sum_{s\geq 0}\binom{-n}{s}\left(-\frac{x^{k-1}}{k!}\right)^s \\
&= \frac{\ell}{n}[x^{n-\ell}]\sum_{s\geq 0}\binom{n+s-1}{s}\frac{x^{s(k-1)}}{k!^s}.
\end{aligned}$$

So, setting $n-\ell = s(k-1)$, we have $n = s(k-1)+\ell$, and the last displayed chain of equalities implies that

$$[x^n]T_k^\ell(x) = \frac{\ell}{s(k-1)+\ell}\binom{ks+\ell-1}{s}\frac{1}{k!^s}. \tag{5.21}$$

Note that in particular, for $\ell = 1$, we get

$$[x^n]T_k(x) = \frac{1}{s(k-1)+1}\binom{ks}{s}\frac{1}{k!^s}. \tag{5.22}$$

The previous example was about nonplane trees, and used exponential generating functions. The next example shows that the Lagrange Inversion Formula can be used in situations in which ordinary generating functions appear.

**Example 5.64** *How many plane rooted unlabeled trees are there on $n$ vertices in which every vertex has either 0 or $k$ children?*

**Solution:** Let $U_k(x)$ be the ordinary generating function counting such trees. That is, $U_k(x) = \sum_{n\geq 1} u_{n,k}x^n$, where $u_{n,k}$ is the number of such trees on $n$ vertices. Then

$$U_k(x) = x + xU_k(x)^k = x\left(1 + U_k(x)^k\right), \tag{5.23}$$

since removing the root of such a tree that consists of more than one vertex results in a $k$-element sequence of such trees.

Rearranging (5.23), we get

$$\frac{U_k(x)}{1 + U_k(x)^k} = x,$$

so

$$U_k(x) = \left(\frac{x}{1+x^k}\right)^{\langle -1\rangle}.$$

Setting $\ell = 1$, the Lagrange Inversion Formula yields

$$\begin{aligned}[x^n]U_k(x) &= \frac{1}{n}[x^{n-1}]\left(\frac{x}{(x/(1+x^k))}\right)^n \\ &= \frac{1}{n}[x^{n-1}]\left(1+x^k\right)^n \\ &= \frac{1}{n}[x^{n-1}]\sum_{s\geq 0}\binom{n}{s}\left(x^k\right)^s \\ &= \frac{1}{n}[x^{n-1}]\sum_{s\geq 0}\binom{n}{s}x^{ks}.\end{aligned}$$

So $u_{n,k} = 0$ if $n-1$ is not divisible by $k$, and

$$u_{n,k} = \frac{1}{n}\binom{n}{s} \tag{5.24}$$

if $n = ks+1$.

Note that for $k = 2$, this means that there are no such trees on $n$ vertices if $n$ is odd, and there are $\frac{\binom{2m+1}{m}}{2m+1} = \frac{\binom{2m}{m}}{m+1}$ such trees if $n = 2m+1$. Compare this result with Theorem 5.28. ◊

When we discussed Theorem 5.55, we were not yet able to deduce a formula for the number of all rooted trees on vertex set $[n]$ from the equality $T(x) = xe^{T(x)}$. Now, with the Lagrange Inversion Formula at hand, we can achieve that.

**Example 5.65** *The number of rooted trees on vertex set* $[n]$ *is* $n^{n-1}$.

**Solution:** Let $t_n$ be the number of rooted trees on $n$ vertices, with $t_0 = 0$, and let $T(x) = \sum_{n\geq 1} t_n \frac{x^n}{n!}$ be the exponential generating function of the sequence of the numbers $t_n$. As we mentioned in the proof of Theorem 5.55, a rooted tree naturally decomposes to its root (that has generating function $x$) and a set of rooted trees (that has generating function $e^{T(x)}$.) Therefore, the Product Formula yields

$$T(x) = xe^{T(x)}. \tag{5.25}$$

So $T(x)/e^{T(x)} = x$, and therefore,

$$T(x) = \left(\frac{x}{e^x}\right)^{\langle -1\rangle}.$$

Setting $\ell = 1$, the Lagrange Inversion Formula yields

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[x^{n-1}]e^{nx} \\
&= \frac{1}{n}[x^{n-1}]\sum_{m\geq 0}\frac{(nx)^m}{m!} \\
&= \frac{1}{n}\frac{n^{n-1}}{(n-1)!} \\
&= \frac{1}{n}\frac{n^n}{n!} \\
&= \frac{n^{n-1}}{n!}.
\end{aligned}$$

So $t_n = n^{n-1}$ as claimed. ◊

Quick Check Exercises 1 and 2, as well as Supplementary Exercise 39 cover applications of the Lagrange Inversion Formula with $\ell > 1$.

## Quick Check

1. Use the Lagrange Inversion Formula to compute the number of rooted non-plane forests with two components on vertex set $[n]$.
2. Let $k \geq 2$ be a fixed integer. Use the Lagrange Inversion Formula to compute the number of ordered pairs of plane rooted unlabeled trees in which each vertex has either $k$ or zero children.
3. Prove that the following statement is equivalent to the Lagrange Inversion Formula. If $x = f(x)/G(f(x))$, where $G$ is a formal power series with a nonzero constant term, while $f$ is a formal power series with a zero constant term, then

$$n[x^n]f(x)^k = k[x^{n-k}]G(x)^n. \tag{5.26}$$