(b) Let $B(x) = 1 + \sum_{n\geq 1} 2^{\binom{n-1}{2}} \frac{x^n}{n!}$, and let

$$Eu(x) = \sum_{n\geq 1} Eu(n)\frac{x^n}{n!},$$

where $Eu(n)$ denotes the number of Eulerian graphs on vertex set $[n]$. Then

$$Eu(x) = \ln(B(x)).$$

5.9 Exercises

1. Prove that any finite simple graph with at least two vertices has two vertices that have the same degree.
2. Let G be any finite graph, and let x and y be two vertices of G. Prove that if there is a walk in G from x to y, then there is also a path from x to y in G.
3. Prove that the finite simple graph G is a tree if and only if, for any two distinct vertices x and y, there is exactly one path connecting x to y.
4. Let T be a tree, let P be a path in T, and let v be a vertex of T outside P. Prove that there is a vertex $w \in P$ that is closer to v than any other vertices of P.
5. Prove that there are at least 25 trees on 10 vertices so that no two of them are isomorphic.
6. + We say that the parking function f has no *like consecutive elements* if there is no i in the domain of f so that $f(i) = f(i+1)$. Find a formula for the number of parking functions on $[n]$ that have no like consecutive elements.
7. Let $f : [n] \rightarrow [n]$ be a parking function. Prove that the number of unsuccessful parking attempts in f is equal to $\binom{n+1}{2} - \sum_{j=1}^{n} f(j)$. An unsuccessful attempt is when a car tries to park in a spot but finds another car there.
8. Let us order the elements of the degree sequence of a graph in nonincreasing order and call the obtained sequence the *ordered degree sequence* of the graph.

   How many ordered degree sequences are possible for a tree on n vertices? (Your answer can contain the function p denoting the number of partitions of an integer.)

9. + Let us generalize the notion of parking functions as follows: Let $1 \leq k \leq n$. We now have $n+1-k$ cars arriving at our parking lot, which still has n spots labeled 1 through n. Each car has a favorite spot, and the preferences are described by a function $f : [n+1-k] \rightarrow [n]$. One day, however, spots labeled less than k are closed for construction. If all of the $n+1-k$ cars can park using our usual parking process, then we say that f is a *k-shortened parking function* (a term coined by Catherine Yan). Note that 1-shortened parking functions are simply parking functions.

   (a) Find a sufficient and necessary condition for f being a k-shortened parking function.

   (b) Find a formula for the number of k-shortened parking functions $f : [n+1-k] \rightarrow [n]$.

10. Find a formula for the number $P(n,k)$ of parking functions on $[n]$ that contain exactly k values equal to 1.

11. + Let us call a parking function on $[n]$ *prime* if, by omitting a 1 from the parking function, we get a parking function on $[n-1]$. For instance, $1,1,2$ is a prime parking function, but $1,1,3$ is not. By convention, we say that the only parking function on $[1]$ is also prime.

    The rest of this exercise is meant to explain the name "prime parking functions" by showing how each parking function can be decomposed uniquely into a sequence of prime parking functions, the sum of whose sizes is n.

    To that end, prove that there exists a natural bijection from the set $\mathbf{P}(n)$ of all parking functions on $[n]$ onto the set $SPP(n)$ of strings

    $$(p_1, p_2, \cdots, p_k, s_1, s_2, \cdots, s_k),$$

    where k ranges from 1 through n (so k is not fixed), each p_i is a prime parking function on $[a_i]$, and $\sum_{i=1}^{k} a_i = n$. Finally, $(s_1, s_2, \cdots, s_n)$ is a partition of the set $[n]$ so that block s_i has a_i elements. In other words, the bijection we are looking for will not only specify the prime parking functions into which our parking function is decomposed, but also their *location* within the original parking function.

12. Let $P(n,k)$ be the number of parking functions on $[n]$ containing exactly k values equal to 1, and let $PP(n,k)$ be the number of prime parking functions containing exactly k values equal to 1. Express $PP(n+1,k+1)$ by $P(n,k)$.

13. + Prove that we have $PP(n) = (n-1)^{n-1}$ for all $n \geq 1$.

14. Prove that

    $$\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!} = \frac{1}{1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}}.$$

15. Find the number $u(n)$ of different unlabeled trees on n vertices for all $n \leq 6$.
16. For each of the three graphs shown in Figure 5.46, find the number of all automorphisms of that graph.

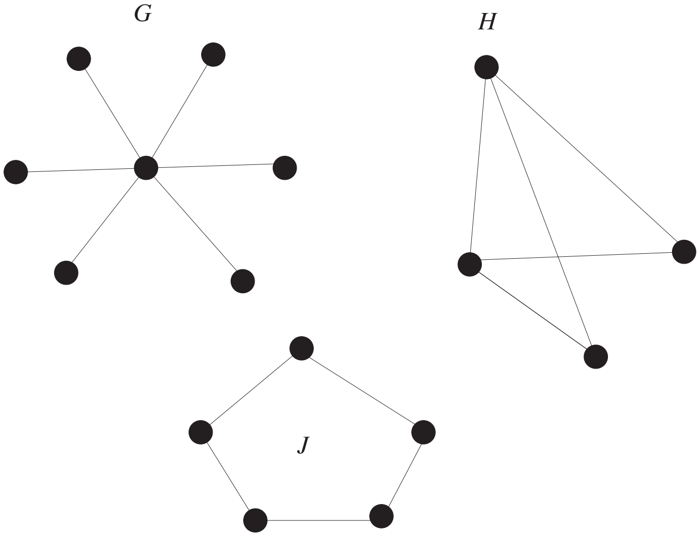

Figure 5.46
How many automorphisms do these graphs have?

17. Show an example of a graph on six vertices that has 720 labelings.
18. Let T be a rooted tree on $n+1$ unlabeled vertices. Let us now label the vertices of T by the elements of $[n+1]$ so that each label is used exactly once and the label of each vertex is smaller than the label of its parent. See Figure 5.47 for an example. We will say that trees labeled in this way are *decreasing* trees. Prove that the number of

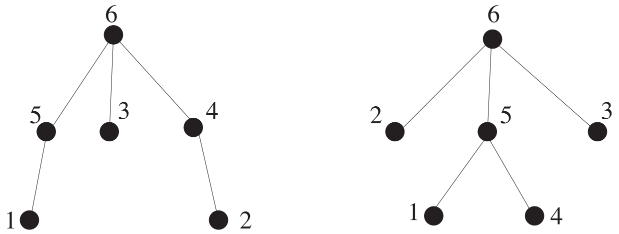

Figure 5.47
Decreasing trees on six vertices.

decreasing trees on $n+1$ vertices is $n!$.

19. Find a formula for the number of increasing trees on $n+1$ vertices in which the root has exactly k children.

20. (a) Find a bijection f from the set of 132-avoiding n-permutations, defined in Exercise 32 of Chapter 4, onto that of rooted plane trees on $n+1$ unlabeled vertices.

    (b) Give a bijection f as an answer to part (a) that turns the number of left-to-right minima of a permutation p into the number of leaves of the tree $f(p)$.

21. Pattern-avoiding permutations were defined in Exercise 32 of Chapter 4. Find a bijection from the set of all unlabeled binary plane trees on n vertices onto the set of all 132-avoiding n-permutations.

22. Use the result of the previous exercise to prove that the number of 132-avoiding n-permutations with k descents is the same as the number of 132-avoiding n-permutations with k ascents.

23. Compare the results of Exercises 20 and 22 and prove Theorem 5.29.

24. What is the number $a_{n,k}$ of acyclic functions on $[n]$ that have k fixed points?

25. Let $a_{n,k}$ be defined as in the previous exercise. Find a simple closed form for the generating function

$$A_n(x) = \sum_{k=1}^{n} a_{n,k} x^k.$$

26. Prove that, for all graphs G, the number of pairs (g, A) in which

    (a) A is an acyclic orientation of G, and

    (b) g is a coloring of the vertices of G using only colors from $[n]$ so that $g(u) > g(v)$ if there is an edge from u to v in A

    is equal to $\chi_G(n)$.

27. Let n and m be two integers with $n < m$, so that both n and m are roots of χ_G. Prove that, in that case, all integers in the interval $[n, m]$ are roots of χ_G.

28. + Prove that, if two graphs have identical chromatic polynomials, then they must have the same number of edges.

29. Let Co_n be the number of connected graphs on vertex set $[n]$.

    (a) Prove that the number of *rooted* graphs on vertex set $[n]$ is

$$\sum_{k=1}^{n} k \binom{n}{k} Co_k \cdot 2^{\binom{n-k}{2}}.$$

    A rooted graph is a graph in which one vertex is designated as the root.