Quick Check

1. Let $\mathcal{H}$ be a hypergraph on vertex set $[n]$ in which no two edges are disjoint. Is it true that we can add edges to $\mathcal{H}$ until we reach a total of 2^{n-1} edges so that the obtained hypergraph still does not have two disjoint edges?
2. Let $\mathcal{H}$ be a hypergraph on vertex set $[n]$ so that for any two edges A and B of $\mathcal{H}$, the inequality $|A \cap B| \geq 2$ holds. Prove that $\mathcal{H}$ has at most 2^{n-2} edges.
3. Matchings in graphs are defined in Exercise 34 of Chapter 5. Let G be a simple graph with more than 200 edges. Is it true that if G does not have a vertex of degree at least 11, then G must contain a matching consisting of at least 11 edges?

6.3 Something is more than nothing: Existence proofs

This section is built on the powerful, if not earth-shattering, idea that if the number of certain objects is positive, then it is at least one, so at least one such an object *exists*. Before the reader laughs at us for talking about such trivialities, we mention that we will use this method in situations when the existence of certain constructions is highly nontrivial and therefore needs to be proved.

6.3.1 Property B

As hypergraphs are still fresh in our minds, we start with an example from that area. Property B is a well-studied property of hypergraphs. We explain it by the following example.

Example 6.41 *A football coach wants to try out various offensive formations with his players. The coach has n offensive players, and the day before practice, he designs m offensive formations, each of which will consist of k offensive players. Then, at the beginning of the practice, the coach gives an orange or a blue shirt to each player to wear. Prove that, if $m < 2^{k-1}$, then, no matter how the offensive formations have been designed, the coach can assign the shirts so that both colors will be represented in each offensive formation.*

In general, we say that a hypergraph $\mathcal{F}$ on $[n]$ has *property B* (bi-colorable) if it is possible to color the elements of $[n]$ using two colors so that *no* edge of $\mathcal{F}$ is monochromatic. In this terminology, Example 6.41 says that, if a *k-uniform* hypergraph $\mathcal{F}$ on $[n]$ has less than 2^{k-1} edges, then $\mathcal{F}$ has property B.