3. Sperner's theorem. The number of edges in a hypergraph $\mathcal{F}$ on $[n]$ in which no edge contains another edge is

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

(C) Codes over finite alphabets

1. Kraft's inequality. A prefix-free code over $[k]$ with N_j words of length j exists if and only if

$$\sum_{j \geq 1} \frac{N_j}{k^j} \leq 1. \tag{6.13}$$

2. MacMillan's theorem. A uniquely decipherable code over $[k]$ with N_j words of length j exists if and only if (6.13) holds.

6.6 Exercises

Remember that, unless otherwise stated, all graphs mentioned are *simple graphs*.

1. Let G be a graph on $2n$ vertices and having more than n^2 edges. Prove by induction that G contains a triangle.
2. Let $\delta(A)$ (resp. $e(A)$) denote the minimum degree (resp. number of edges) of the graph A, and let $d_A(x)$ denote the degree of the vertex x in the graph A.
   (a) Prove that, any graph G contains a bipartite subgraph B so that $e(B) \geq e(G)/2$.
   (b) Prove that any graph G contains a bipartite subgraph B so that the vertex sets of B and G are the same and $\delta(B) \geq \delta(G)/2$.
   (c) Prove that any graph G contains a bipartite subgraph B so that the vertex sets of B and G are the same and so that for all vertices $x \in B$, the inequality $d_B(x) \geq d_G(x)/2$ holds.
3. Prove that for all nonnegative integers n, the function $f(x) = \binom{x}{r}$ is convex in the interval $[r-1, \infty)$.
4. Let G be a graph, and let G_1 be the graph obtained by taking two copies of G, then connecting each vertex x of G to the corresponding vertex x' of G_1. See Figure 6.13 for an illustration. Express $\chi(G_1)$ in terms of $\chi(G)$.

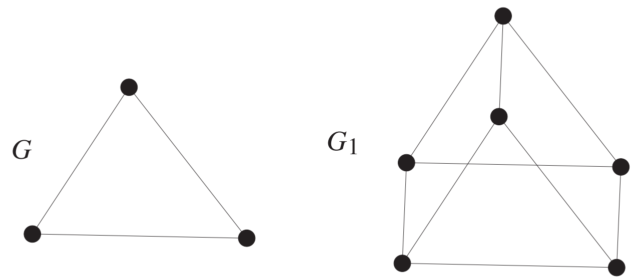

Figure 6.13
An example of the graphs G and G_1.

5. Let G and H be two simple graphs, and define the *weak direct product* $G \times H$ of G and H as follows: The vertex set of $G \times H$ is the set of all ordered pairs (g, h), where g is a vertex of G and h is a vertex of H. There is an edge between (g, h) and (g', h') if there is an edge between g and g' and there is an edge between h and h'. Figure 6.14 shows the weak direct product of K_2 and K_3.

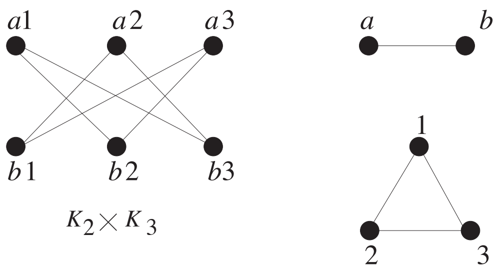

Figure 6.14
The weak direct product of K_2 and K_3.

   (a) Prove that $\chi(G \times H) \leq \min(\chi(G), \chi(H))$.
   (b) Is it true that $\chi(G \times G) = \chi(G)$?

6. A graph G on 10 vertices has degrees 9, 3, 3, 3, and six degrees equal to 2. Prove that $\chi(G) \leq 4$.
7. Prove that, if G is as in Exercise 6, then $\chi(G)$ can be 3 or 4, but not 2.
8. A connected graph is called *k-connected*, or k-vertex-connected, if it stays connected even if we remove any $(k - 1)$ of its vertices and all edges adjacent to them.

At most how many edges can a simple graph on n vertices have if it is not 2-connected?

9. Let G be a graph with maximum degree k.
   (a) Prove that $\chi(G) \leq k+1$.
   (b) Find two infinite classes of graphs for which the bound of part (a) is sharp, that is, $\chi(G) = k+1$.
10. + Let G be a graph with maximum degree k.
    (a) Find *all* graphs G that are 3-connected and for which $\chi(G) = k+1$.
    (b) Find all graphs G for which $\chi(G) = k+1$.
11. The set of workers at a factory altogether have 500 different skills, and for each skill, there are 10 workers who have that skill. Prove that it is possible to schedule the workers in two shifts so that each skill is available in both shifts.
12. Let $\mathcal{F}$ be a hypergraph on $[n]$ so that, for any two edges A and B in $\mathcal{F}$, the inequality $A \cup B \neq [n]$ holds. At most how large can $\mathcal{F}$ be?
13. The proof we gave for Theorem 6.30 showed an example for $\mathcal{F}$ that contained edges of $[n]$ of all sizes from 1 to n. Find an alternative proof that uses edges of a smaller number of different sizes.
14. Let $\mathcal{F}$ be a hypergraph on $[n]$ so that the smallest edge in $\mathcal{F}$ is of size k. Assume that, no matter how we choose $k+1$ edges in $\mathcal{F}$, their intersection is never empty. Can the intersection of *all* edges in $\mathcal{F}$ be empty?
15. + Show an example of a 2-coloring of the set [2141] that does not contain a monochromatic arithmetic progression of length 18.
16. We color each edge of K_6 either red or blue. Prove that the resulting graph contains a triangle with monochromatic edges.
17. + Let k and ℓ be positive integers. Prove that there exists a smallest positive integer $R(k,\ell)$ so that, if we color the edges of $K_{R(k,\ell)}$ red or blue, there will be either a red copy of K_k or a blue copy of K_ℓ.

    Note that $R(k,\ell)$ is called a *Ramsey number*.
18. We color each edge of K_{17} either red or blue or green. Prove that the resulting graph contains a triangle with monochromatic edges.
19. Prove that a code over a finite alphabet is instantaneous if and only if it is prefix-free.
20. Having defined the notion of avoidance for graphs and permutations, let us define avoidance for *matrices whose entries are either 0 or 1* as follows: We say that the $n \times n$ matrix A *avoids* the $k \times l$ matrix B if we cannot find k rows and l columns in A which is

entry-wise at least as large as B. In other words, A does not have a $k \times l$ submatrix that has a 1 in each position where B has a 1.

Let $f(n, B)$ be the highest number of entries equal to 1 that a B-avoiding $n \times n$ matrix having 0s and 1s for entries can have.

(a) Find $f(n, B)$ if $B = (1, 1)$.

(b) + Find $f(n, B)$ if $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

21. The previous exercise might have the reader guess that $f(n, B) = O(n)$ for all matrices B. This is not true, however, as we will see in this exercise.

    Let
    $$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

    Let the sequence of matrices A_i be defined by $A_0 = 1$, $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $A_{n+1} = \begin{pmatrix} I_{2^n} & A_n \\ A_n & 0 \end{pmatrix}$, where I_{2^n} is the identity matrix of size $2^n \times 2^n$. So A_n is of size $2^{n+1} \times 2^{n+1}$.

    (a) Prove that A_n avoids B.

    (b) Determine how many entries equal to 1 the matrix A_n contains, and conclude that $f(n, B) \neq O(n)$.

22. Let B be a *permutation matrix* of size $k \times k$, and assume that n is divisible by k^2. Let A be a B-avoiding $n \times n$ matrix whose entries are equal to 0 or 1, so that A has $f(n, B)$ entries equal to 1. Cut A up into blocks of size $k^2 \times k^2$. Show that at most $f\left(\frac{n}{k^2}, B\right)$ of these blocks can contain nonzero entries.

23. + Continuing the previous exercise, let us call a $k^2 \times k^2$ block of A *tall* if it contains nonzero entries in at least k rows. Similarly, call a block of A *wide* if it contains nonzero entries in at least k columns.

    Keeping in mind that A and B are as defined in the previous exercise, look at any row of blocks of A. Prove that at most $\binom{k^2}{k}(k-1)$ of these n/k^2 blocks can be tall. Formulate and prove the corresponding statement for wide blocks.

24. Use the results of the previous two exercises to prove that, if B is a permutation matrix, then
    $$f(n, B) \leq 2k^4 \binom{k^2}{k} n.$$

    Note in particular that this means that, if B is a permutation matrix, then $f(n, B) = O(n)$.

25. + Recall that $S_n(q)$ is the number of n-permutations avoiding the pattern q. This notion was defined in Exercise 32 of Chapter 4. Use the result of the previous exercise to show that, for all patterns q, there exists a constant c_q so that $S_n(q) < c_q^n$.

26. Let n be a positive integer, and let q be a k-permutation, with $k \leq n$. Prove that there exists an n-permutation containing more than $\binom{n}{k}/k!$ copies of q.

27. A company has n job openings to fill, and n applicants. Each candidate has his own set of qualifications and demands. That is, a candidate may qualify for certain jobs but not for others, and each candidate has a minimum salary expectation for each job he is interested in, meaning that he will not take the job for a salary lower than that expected amount. These amounts vary from candidate to candidate, and from job to job. Nobody can work in more than one job.

    A Human Resources manager wants to fill all n positions so that the total salary cost for the company is as low as possible.

    (a) Express the task of the manager in the language of graphs.
    (b) Assume the manager fills the openings in a greedy way. That is, he first fills the opening for which he finds a qualified candidate at the lowest possible salary, then, from the remaining $n-1$ openings, he first fills the one for which he finds a qualified candidate at the lowest possible salary, and so on. Assuming that the manager succeeds in filling all positions, will he always achieve the lowest possible cost using this strategy?

28. + Let A and B be two cycle-free subgraphs of the same graph K, and assume that A has more edges than B. Prove that there is an edge of A that can be added to B so that B keeps its cycle-free property.

29. + A county wants to build a connected system of roads for its n towns. A case study revealed the costs of building a direct road between each pair of towns in the county. The county commissioner knows that the cheapest connected network will be achieved if the graph G of roads to be built is a tree. Therefore, the commissioner plans to build G in the following greedy way: First, she picks the cheapest road possible between two towns. Then she picks the second cheapest road. In each following step, she picks the cheapest road still not picked that will not create a cycle in the graph of roads already picked. She will stop when it is no longer possible to pick a road without creating a cycle, that is, when the graph G built is a tree.

    Will this greedy strategy always provide the best possible results?