7

Analytic combinatorics

It often occurs in enumerative combinatorics that obtaining an *exact formula* that answers a question is difficult, or time-consuming, or even impossible, while it is much easier to obtain a formula that approximates the exact solution up to a specified level of precision. In this chapter, we consider examples of this phenomenon, and methods to apply in these situations. Unless otherwise noted, all power series in this chapter are assumed to have *complex* coefficients.

7.1 Exponential growth rates

As we will see, it is often reasonably easy to prove that a sequence grows *roughly* at the speed of the powers of a fixed positive real number. We start our excursion to analytic combinatorics at such sequences.

7.1.1 Rational functions

Rational functions, which are the ratio of two polynomials, are a good starting point to learn our methods.

7.1.1.1 Motivation

What is larger, the number of compositions of 1000 into parts that are equal to 1 or 4, or the number of compositions of 1000 that are equal to 2 or 3?

If a_n denotes the number of the former, and b_n denotes the number of the latter, then it is easy to prove by the methods of Chapter 3 that

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{1}{1-x-x^4}, \tag{7.1}$$

and

$$B(x) = \sum_{n\geq 0} b_n x^n = \frac{1}{1-x^2-x^3}. \tag{7.2}$$

In theory, nothing prevents us from finding the partial fraction decompositions of $A(x)$ and $B(x)$, deducing exact formulas for a_n and b_n, and then

DOI: 10.1201/9781003304272-7

comparing the values of a_{1000} and b_{1000}. However, in practice, that procedure would be exceedingly cumbersome. The partial fraction decompositions of $A(x)$ and $B(x)$ would contain fractions with complex numbers, and the exact formulas obtained for a_n and b_n would not be particularly pleasant. Comparing the values for $n = 1000$ could pose another problem.

It is therefore a natural question to ask if we can proceed in a simpler way. Is there a way to decide which of the two sequences will grow faster simply by perusing their generating functions, and not computing the actual numbers a_n and b_n?

Fortunately, the answer to that question is in the affirmative. In this section we develop the tools to see why and how.

7.1.1.2 Theoretical background

The following definition will be our crucial tool in describing how fast a given sequence grows.

Definition 7.1 *Let $s_0, s_1, s_2, \ldots$ be an infinite sequence of complex numbers. Let α be a real number. We say that the sequence has* exponential growth rate *α if*

> *the sequence of the numbers s_n does not have an infinite subsequence $r_1, r_2, \cdots$ for which*
>
> $$\lim_{n\geq\infty} \sqrt[n]{|r_n|} > \alpha,$$
>
> *and*
>
> *the sequence of the numbers s_n does have an infinite subsequence $r_1, r_2, \cdots$ for which*
>
> $$\lim_{n\geq\infty} \sqrt[n]{|r_n|} = \alpha.$$

In other words, α is the *largest number* so that there is an infinite subsequence of the sequence $s_1, s_2, \cdots$, (or, in what follows, the sequence s_n) for which the limit of the absolute values of the nth roots of the elements is α. Yet in other words, $\alpha = \limsup_{n\geq 1} \sqrt[n]{|s_n|}$.

Example 7.2 *Let $s_n = n \cdot 2^n$. Then the sequence of the numbers s_n has exponential growth rate 2.*

Solution: This follows from the fact that $\sqrt[n]{s_n} = 2\sqrt[n]{n} \rightarrow 2$. ♢

Example 7.3 *Let $s_n = 2^n/n^{100}$. Then the sequence of the numbers s_n has exponential growth rate 2.*

Solution: This follows from the fact that $\sqrt[n]{s_n} = 2/\sqrt[n]{n^{100}} \rightarrow 2$. ♢

The two examples above show that two sequences can differ by a large polynomial factor and still have the same exponential growth rate. This is because polynomial factors are much smaller than exponential ones. Indeed, as $\lim_{n\to\infty} \sqrt[n]{|p(n)|} = 1$ for any nonzero polynomial $p(n)$, multiplying or dividing a sequence by any nonzero polynomial will not change its growth rate. Note that it follows that, if $f_n = p(n)$ for some nonzero polynomial p, then the exponential order of the sequence f_n is 1.

Example 7.4 *Let $s_n = 3^n + (-3)^n$. Then the sequence of the numbers s_n has exponential growth rate 3.*

Note that, in the last example, every other term is 0, but that does not affect the exponential growth rate of the sequence. It is the infinite subsequence with the highest growth rate that provides the growth rate of the whole sequence. In this case, that is the subsequence $r_n = s_{2n}$.

It goes without saying that not all sequences have a finite growth rate. Recall that the notation $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$. As sequences are functions whose domain is the set of natural numbers, this definition is meaningful for sequences as well.

Example 7.5 *Let $a_n = n!$. Then the sequence a_n has an infinite growth rate.*

Solution: This can be proved by Stirling's formula, which states that $n! \sim (n/e)^n\sqrt{2\pi n}$, so $\sqrt{a_n} \sim n/e$, which clearly diverges to infinity. We first mentioned this formula in (1.13). Alternatively, one can use the fact that $a_{n+1}/a_n = n + 1$, which also diverges to infinity. ♢

This is not to say that if $a_n \geq n!$ holds for the elements of a sequence, then we cannot ask interesting questions about the sequence that involve exponential growth rate. For instance, we can ask *how much faster* the sequence grows than the sequence of factorials, by asking for the exponential growth rate of the sequence $a_n/n!$. This will be a particularly appropriate question in cases when the exponential generating function of the sequence a_n can be computed.

The reader is asked to prove the following propositions in Supplementary Exercise 2.

Proposition 7.6 *Let the sequence f_n have exponential growth rate a, and let the sequence g_n have exponential growth rate b, and let us assume that both sequences consist of complex numbers. Let us assume that $a > b$. Then the sequence $f_n + g_n$ has exponential growth rate a.*

Note that Proposition 7.6 is not true without the condition that $a > b$. A counterexample is $f_n = (-2)^n$ and $g_n = (-2)^{n+1}$. Nevertheless, the following weaker statement always holds.

Proposition 7.7 *Let the sequence f_n have exponential growth rate a, and let the sequence g_n have exponential growth rate b, and let us assume that both sequences consist of complex numbers. Let us assume that $a \geq b$. Then the exponential growth rate of $f_n + g_n$ is at most a.*

Proof: Let us assume the contrary, that is, that the sequence $f_n + g_n$ has an infinite subsequence s_i (where $i \in I$ for some infinite set I of positive integers) that satisfies the inequality $\lim_{i\to\infty} \sqrt[n]{s_i} = a' > a$.

Consider the infinite subsequence $s_i = f_i + g_i$. Let $I_1 \subseteq I$ be the set of indices for which $|f_i| \geq |g_i|$, and let $I_2 \subseteq$ be the set of indices for which $|f_i| < |g_i|$. As I is infinite, so is at least one of I_1 and I_2. Let us assume without loss of generality that I_1 is infinite.

Now notice that, if $i \in I_1$, then

$$|s_i| = |f_i + g_i| \leq |f_i| + |g_i| \leq 2|f_i|,$$

which implies that the sequence f_i (where $i \in I_1$) has exponential order at least a', contradicting our hypothesis. ◊

Now we start putting the growth rate of a sequence in the context of generating functions. Note that, unless otherwise stated, all power series in this section are taken over the field of *complex numbers*, that is, they have complex coefficients. We are only assuming that the reader is familiar with both ways of writing a complex number z, namely, the Cartesian way, which sets $z = x+iy$, where x and y are real numbers, and the polar coordinate way, which sets $z = re^{i\phi}$, where r is a nonnegative real number and ϕ, called the *argument* of z, is an angle measured in radians. The reader is probably familiar with basic complex arithmetics, that is, the reader knows how to add, subtract, multiply, and divide complex numbers, how to take their powers, and how to compute their absolute values. We also assume that the reader is familiar with Euler's formula, stating that

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{7.3}$$

for all real numbers ϕ.

Recall that a rational function is the ratio of two polynomials.

Definition 7.8 *Let $F(x) = P(x)/Q(x)$ be a rational function so that $P(x)$ and $Q(x)$ do not have any roots in common. Then the complex number x_0 is a* singular point *of $F(x)$ if $Q(x_0) = 0$.*

Example 7.9 *The rational function*

$$F(x) = \frac{x^2 + 3x + 2}{(x-1)^2(x^2+1)}$$

has singular points $x_0 = 1$, $x_0 = i$, and $x_0 = -i$.

Singular points of a rational function written in simplest terms are just the roots of the denominator. Still, the name singular point is worth introducing since it will have a meaning for functions that are not rational functions as well.

If r_1 is a root of the polynomial $Q(x)$, then we say that r_1 is a root of *smallest modulus* of $Q(x)$ if there is no root r_2 of $Q(x)$ that satisfies $|r_1| > |r_2|$.

The following theorem describes an easy way to find the exponential growth rate of the sequence of coefficients of a rational function.

Theorem 7.10 *Let*

$$S(x) = \sum_{n\geq 0} s_n x^n = \frac{P(x)}{Q(x)}$$

be a rational function with $Q(0) \neq 0$, let us assume that $P(x)$ and $Q(x)$ do not have any roots in common, and let us also assume that $Q(x)$ has a unique *root r_1 of smallest modulus.*

Then the exponential growth rate of the sequence of the coefficients s_n is equal to $|z_1|$, where $z_1 = 1/r_1$.

Note that the theorem holds even without the assumption that $Q(x)$ has a unique root of smallest modulus, but it would be a bit cumbersome to prove that with our current tools. We will prove that version of the theorem in the next section.

Proof: We can assume without loss of generality that $Q(x)$ has constant term 1. Indeed, dividing a power series by a constant does not change the exponential growth rate of its coefficients.

Now let $Q(x) = (1-z_1x)^{d_1}(1-z_2x)^{d_2}\cdots(1-z_kx)^{d_k}$. In other words, let d_i be the multiplicity of r_i as a root of $Q(x)$. Note that this equation is equivalent to $Q(x) = C(r_1 - x)^{d_1}(r_2 - x)^{d_2}\cdots(r_k - x)^{d_k}$.

We can assume that the degree of $Q(x)$ is larger than that of $P(x)$. The reader is asked to prove this simple fact in Exercise 1.

Then, just as the reader has seen in calculus, $S(x) = P(x)/Q(x)$ has a partial fraction decomposition of the form

$$\begin{aligned} S(x) &= \frac{C_{1,1}}{1-z_1x} + \frac{C_{1,2}}{(1-z_1x)^2} + \ldots + \frac{C_{1,d_1}}{(1-z_1x)^{d_1}} + & (7.4)\\ &+ \frac{C_{2,1}}{1-z_2x} + \frac{C_{2,2}}{(1-z_2x)^2} + \ldots + \frac{C_{2,d_2}}{(1-z_2x)^{d_2}} + \ldots & (7.5)\\ &+ \frac{C_{k,1}}{1-z_kx} + \frac{C_{k,2}}{(1-z_kx)^2} + \ldots + \frac{C_{k,d_k}}{(1-z_kx)^{d_k}}, & (7.6)\end{aligned}$$

where the $C_{i,j}$ are constants.

Indeed, multiplying both sides by $Q(x)$, we get an equality involving two polynomials, one of which is obviously the polynomial $P(x)$. Equating coefficients of x^n for all n, we can determine the values of each of the $C_{i,j}$.

So $S(x)$ is a sum of a few power series, each of which is of the form $C_{i,j}/(1-z_ix)^j$. Which of these summands has a coefficient sequence with the highest exponential growth rate? We know from the general version of the binomial theorem (Theorem 3.2) that

$$\frac{1}{(1-z_ix)^j} = (1-z_ix)^{-j} = \sum_{n\geq 0} z_i^n \binom{j+n-1}{j-1} x^n.$$

As $\binom{j+n-1}{j-1}$ is just a polynomial of degree $j-1$ in n, it follows that the exponential growth rate of the power series of this type is $|z_i|$, for *any* j.

Therefore, the sequence s_n (the sequence of coefficients of $S(x)$) is the sum of various sequences whose exponential growth rates $|z_i|$ we know, namely,

$$s_n = \sum_{i=1}^{k} \sum_{j=1}^{d_j} C_{i,j} z_i^n \binom{j+n-1}{j-1}.$$

Recall that r_1 is the only root of $Q(x)$ that is of smallest modulus, so $|z_1| > |z_i|$ for any $i \neq 1$. Therefore, the sum g_n of all these sequences that do not involve z_1 (those for which $i \neq 1$) has an exponential growth rate less than $|z_1|$ by Proposition 7.7. On the other hand, the sum f_n of sequences that do involve z_1 has exponential order $|z_1|$ since it simply equals z_1^n times a nonzero polynomial. Then our claim follows by Proposition 7.6. $\diamondsuit$

It is high time that we returned to the combinatorial example that we mentioned at the beginning of Section 7.1.1.1. That question involved deciding whether the coefficients of $A(x) = 1/(1-x-x^4)$ or the coefficients of $B(x) = 1/(1-x^2-x^3)$ grow faster. Theorem 7.10 tells us that, to find these growth rates, we have to find the roots of smallest modulus of the two denominators. Using our favorite software package, we find that, for $A(x)$, that root is about $r_1 = 0.7245$, whereas for $B(x)$, it is about $r_1 = 0.7549$. Therefore, the exponential growth rate of the sequence a_n is equal to $z_1 = 1/0.7245 = 1.3803$, whereas the exponential growth rate of the sequence b_n is $z_1 = 1/0.7549 = 1.3247$, so the sequence of the numbers a_n grows faster.

Let us consider further examples of finding the exponential growth rate of a combinatorially defined sequence.

Example 7.11 *Let h_n be the number of words of length n over the alphabet $\{A,B,C\}$ that do not contain the subword ABC in consecutive positions. Find the exponential growth rate of the sequence of the numbers h_n.*

Solution: If we take a word w counted by h_n and add one letter to its end, we get a word counted by h_{n+1}, except when w ends in AB and the letter we add to the end of w is C. This leads to the recurrence relation

$$h_{n+1} = 3h_n - h_{n-2}$$

for $n \geq 2$, while $h_0 = 1$, $h_1 = 3$, and $h_2 = 9$. Setting $H(x) = \sum_{n\geq 0} h_n x^n$, the displayed recurrence relation leads to the functional equation

$$H(x) - 9x^2 - 3x - 1 = 3x(H(x) - 3x - 1) - x^3 H(x),$$

$$H(x) = \frac{1}{1 - 3x + x^3}.$$

As any mathematical software package tells us, the root of smallest modulus of the denominator of $H(x)$ is at $x_1 = 0.3473$, so the exponential growth rate of the sequence of the numbers h_n is $1/0.3473 = 2.8794$. ♢

So the exponential growth rate of the sequence h_n is a little bit below 3, which confirms our intuition, since the numbers of *all* words of length n over a 3-element alphabet has exponential growth rate 3.

This is a good time to mention that, in the rest of this chapter, we will often need to find the root of a polynomial $p(x)$ that is closest to 0, for polynomials of degree larger than 2. In such situations, we will simply get that root from a software package.

Example 7.12 *We flip a fair coin n times. Let a_n be the number of possible outcome sequences that do not contain four consecutive heads. Find the exponential order of the sequence of the numbers a_n.*

Solution: We use Theorem 3.25. Let us first consider the acceptable outcome sequences that contain at least one tail. Each such sequence can be broken up into three segments, namely the segment before the last tail, the last tail, and the part after the last tail, which must be all heads, and must be of length at most three.

If an acceptable outcome sequence does not contain a tail, then it is of length at most three, and there is only one such sequence for each such length.

By Theorem 3.25, this leads to the functional equation

$$A(x) = A(x) \cdot x \cdot (1 + x + x^2 + x^3) + 1 + x + x^2 + x^3,$$

$$A(x) = \frac{x^3 + x^2 + x + 1}{1 - x - x^2 - x^3 - x^4}.$$

The root of smallest modulus of the denominator is 0.5188, so the exponential growth rate of the sequence of coefficients of a_n is $1/0.5188 \approx 1.9276$. ♢

We point out that $\lim_{n\to\infty} 1.9276^n/2^n = 0$, so the probability that a sequence of length n of fair coin flips will contain four consecutive heads converges to 1 as n goes to infinity. The exponential growth rate of the number of sequences without four consecutive heads is less than 2 (the exponential growth rate for unrestricted coin flip sequences), but not by much. The reader is invited to explain what happens if we count sequences that do not contain k