Solution: In other words, h_n is the number of ways to partition $[n]$ into any number of blocks of size at most three, then to put an order on the *set* of the blocks. We can use the compositional formula of exponential generating functions to compute $H(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!}$. There is no task to carry out on each block, other than making sure that its size is at most three. This means that $a_k = 1$ if $1 \leq k \leq 3$, and $a_k = 0$ otherwise, so $A(x) = x + \frac{x^2}{2} + \frac{x^3}{3}$. As the blocks are to be linearly ordered, we have $b_m = m!$, so $B(x) = \sum_{m\geq 0} m! \frac{x^m}{m!} = 1/(1-x)$. Therefore, the compositional formula implies

$$H(x) = B(A(x)) = \frac{1}{1 - x - \frac{x^2}{2} - \frac{x^3}{3}}. \tag{7.12}$$

Using a software package, we find that the root of the denominator that is closest to the origin is at $M \sim 0.6725$, so the exponential growth rate of the sequence $h_n/n!$ is $1/M \sim 1.487$. $\Diamond$

Finally, let us take a moment to mention the new, more general version of Theorem 7.10, which is now a special case of Theorem 7.28.

Theorem 7.31 *Let*

$$S(x) = \sum_{n\geq 0} s_n x^n = \frac{P(x)}{Q(x)}$$

be a rational function with $Q(0) \neq 0$, and let us assume that $P(x)$ and $Q(x)$ do not have any roots in common. Let r_1 be a root of $Q(x)$ that is of smallest modulus.

Then the exponential growth rate of the sequence of the coefficients s_n is equal to $|z_1|$, where $z_1 = 1/r_1$.

Note that, in contrast to Theorem 7.10, we no longer need the condition that $Q(x)$ has a *unique* root of smallest modulus.

Quick Check

1. Find the exponential growth rate of the sequence $a_n = n^n/n!$.
2. Find the exponential growth rate of the coefficients of the rational function $f(x) = 1/(4x^2 - 5x + 1)$.
3. Find the exponential growth rate of the coefficients in the power series form of
$$f(x) = \frac{1}{1 - \sqrt{2 - 4x}}.$$

7.2 Polynomial precision

Sometimes it is possible to determine the growth rate of a sequence not only with "exponential," but also with "polynomial precision," and sometimes even more precisely. In this section, we will see examples of this phenomenon.