last session, there will be a break of any (possibly zero) length, the product formula yields

$$S(x) = \sum_{n\geq 0} s_n x^n = \frac{1}{1-x} \cdot \frac{2x}{1-2x} \cdot \frac{x}{1-x} \cdot \frac{2x}{1-2x} \cdot \frac{x}{1-x} \cdot \frac{2x}{1-2x} \cdot \frac{1}{1-x}.$$

Therefore,

$$S(x) = \frac{8x^5}{(1-x)^4(1-2x)^3}.$$

So $r_1 = 1/2$ is a root of multiplicity 3, and $r_2 = 1$ is a root of multiplicity 4. Hence, by Theorem 1, $s_n = p_1(n)2^n + p_2(n)$, where p_1 is a polynomial of degree 2, and p_2 is a polynomial of degree 3. This implies that there exists a constant C so that $s_n \sim Cn^2 2^n$. $\Diamond$

Quick Check

1. Let
$$F(x) = \frac{1}{(1-x)^2(1-2x)}.$$
Find the growth rate of the coefficients of $F(x)$ at polynomial precision.

2. Let
$$G(x) = \frac{1}{(1-4x)^3(1+4x)^3}.$$
Find the growth rate of the coefficients of $G(x)$ at polynomial precision.

3. Let $H(x) = \sum_{n\geq 0} h_n x^n$ so that $h_n \geq 0$ for all n, let us assume that the exponential growth rate of the sequence h_n is at least 1, and let us assume that the sequence h_n is supermultiplicative, that is, the inequality $h_n h_m \leq h_{n+m}$ holds for all n and m. Prove that if H is rational, then it has a *unique* singularity of smallest modulus.

7.3 More precise asymptotics

We can sometimes refine our asymptotic results even further than polynomial precision. We will consider two situations in which this is possible.

7.3.1 Entire functions divided by $(1-x)$

Some power series have a sequence of coefficients that is *convergent* as a sequence, and we can compute the limit of that sequence without computing the coefficients themselves.

Recall that we call a permutation a *derangement* if it does not contain any 1-cycles. Similarly, let us call a permutation a 2-derangement if it does not contain any 1-cycles or 2-cycles. Let $D_{2,n}$ be the number of 2-derangements of length n. What can we say about the ratio $D_{2,n}/n!$? In other words, what is the probability that a randomly selected permutation of length n is a 2-derangement?

Let $D_2(x) = \sum_{n\geq 0} D_{2,n}\frac{x^n}{n!}$ be the exponential generating function of the numbers $D_{2,n}$, where we set $D_{2,0} = 1$. We can obtain $D(x)$ using the exponential formula. Indeed, to get a permutation counted by $D_{2,n}$, we need to partition $[n]$ into blocks of size at least two, then put a k-cycle on each block of size k. There are $(k-1)!$ ways to do that if $k \geq 3$, and 0 ways otherwise. This leads to the generating function

$$\begin{aligned} A(x) &= \sum_{k\geq 3}(k-1)!\frac{x^k}{k!} = \sum_{k\geq 3}\frac{x^k}{k} \\ &= \ln\left(\frac{1}{1-x}\right) - x - \frac{x^2}{2}. \end{aligned}$$

Therefore, by the exponential formula,

$$D(x) = e^{A(x)} = \frac{e^{-x-\frac{x^2}{2}}}{1-x}. \tag{7.24}$$

So, we have an explicit formula for $D(x)$, but how do we use it? Writing up an explicit formula for the numbers $D_{2,n}$ is too complicated, since the numerator of $D(x)$ involves two summation signs, and the division by $1-x$ would mean further summation. It is clear that $D(x)$ has only one singularity, at $x = 1$, but all that means is that the coefficients of $D(x)$, that is, the numbers $D_{2,n}/n!$, have exponential growth rate 1. That is not too surprising, since all sequences that converge to a positive constant have exponential growth rate 1. In order to achieve our goal, that is, to find $\lim_{n\to\infty}\frac{D_{2,n}}{n!}$, we need to learn a new technique. The following theorem is our main tool toward that goal.

Theorem 7.37 *Let f be an entire function, and let*

$$\frac{f(x)}{1-x} = \sum_{n\geq 0} a_n x^n.$$

Then $a_n \sim f(1)$. In other words, $\lim_{n\to\infty} a_n = f(1)$.

Proof: Note that

$$\frac{f(x)}{1-x} = \frac{f(x)-f(1)}{1-x} + \frac{f(1)}{1-x} = \frac{f(x)-f(1)}{1-x} + f(1)\sum_{n\geq 0} x^n. \tag{7.25}$$

Observe that the function $F(x) = \frac{f(x)-f(1)}{1-x}$ is an entire function. Indeed, as $f(x)$ is an entire function, it converges everywhere, so we have $f(x) = \sum_{n\geq 0} f_n x^n$, and, in particular, $f(1) = \sum_n f_n$. Therefore,

$$\begin{aligned} F(x) &= \frac{f(x)-f(1)}{1-x} \\ &= \frac{\sum_{n\geq 0} f_n(x^n-1)}{1-x} \\ &= -\sum_{n\geq 0} f_n\left(1+x+\cdots+x^{n-1}\right) \\ &= \sum_{n\geq 0}\left(-\sum_{k\geq n+1} f_k\right)x^n. \end{aligned}$$

In the last form of $F(x)$, all coefficients are convergent sums, since even the sum $\sum_{k\geq 0} f_k = f(1)$ is finite. For that same reason, the sequence of partial (tail) sums $\sum_{k\geq n+1} f_k$ converges to 0 as n goes to infinity.

In order to prove our claim, let us now compare the coefficient of x^n in the far left and far right terms of (7.25). In the far left, it is, by definition, a_n. The far-right expression is a sum, the second summand of which has coefficient $f(1)$ for each power of x, and the first summand of which has a sequence of coefficients that (as we just proved) converges to 0. $\Diamond$

Corollary 7.38 *As n goes to infinity, the probability that a randomly selected permutation of length n is a 2-derangement converges to $e^{-3/2} \sim$ 0.22313.*

Proof: Just use Theorem 7.37 with $f(x) = e^{-x-x^2/2}$. $\Diamond$

It is also straightforward to recover the result that we proved in Chapter 4, namely, that, for large n, roughly $1/e$ of all permutations of length n are derangements. Indeed, if D_n denotes the number of derangements of length n, then we have seen in (4.11) that

$$\sum_{n\geq 0} D_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}.$$

So using Theorem 7.37 with $f(x) = e^{-x}$, we get that $\frac{D_n}{n!} \approx e^{-1}$.

Example 7.39 *Let h_n be the number of ways to partition $[n]$ into two blocks A and B so that B can be empty, but A cannot, and then to take a partition of A and a permutation of the elements of B. Find $\lim_{n\to\infty} \frac{h_n}{n!}$.*