7.5 Chapter review

1. The exponential growth rate of the sequence $a_0, a_1, \cdots$ is equal to $\limsup \sqrt[n]{|a_n|}$.
2. The exponential growth rate of the sequence of coefficients f_n of a power series $F(x) = \sum_{n\geq 0} f_n x^n$ is equal to $1/M$, where M is the minimal distance between 0 and any singularity of F.
3. If $F(x)$ is a rational function, we can get more precise estimates of the growth rate of the sequence f_n from the *multiplicities* of the singularities of F that are of smallest modulus.

7.6 Exercises

1. Let $P(x)$, $Q(x)$, $A(x)$, and $R(x)$ be polynomials so that $P(x) = Q(x)A(x) + R(x)$, where the degree of $R(x)$ is less than the degree of $Q(x)$. In other words, $R(x)$ is the remainder of $P(x)$ when divided by $Q(x)$.

   Show that the coefficients of the rational functions $P(x)/Q(x)$ and $R(x)/Q(x)$ have the same exponential growth rate.
2. Use an argument from analysis (no software packages) to show that the root r_0 of smallest modulus of $1 - x^k - x^\ell$, where k and ℓ are relatively prime positive integers, is a positive real number.
3. Use an argument from analysis (no software packages) to show that the root r_0 of smallest modulus of the polynomial $Q_a(x) = 1-x-x^4$ is smaller than that of the polynomial $Q_b(x) = 1 - x^2 - x^3$.
4. Let h_n be the number of surjections from $[n]$ to $[k]$ where k varies from 1 to n. Set $h_0 = 1$. Find an explicit formula for $H(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!}$, then find the exponential growth rate of the coefficients $h_n/n!$ of $H(x)$.
5. Explain what the result of the previous exercise means by comparing the numbers of surjections defined on $[n]$ and the number of bijections on $[n]$.
6. Decreasing *nonplane* 1-2 trees were defined in Section 5.5.2.2. For easy reference, such a tree on vertex set $[n]$ is a tree in which each vertex has at most two children, and each child has a label that is less than the label of its parent. The trees are not plane trees, so there is no *left child* or *right child.* Let T_n be the number of such trees. Note that $T_0 = T_1 = T_2 = 1$, while $T_3 = 2$ and $T_4 = 5$.

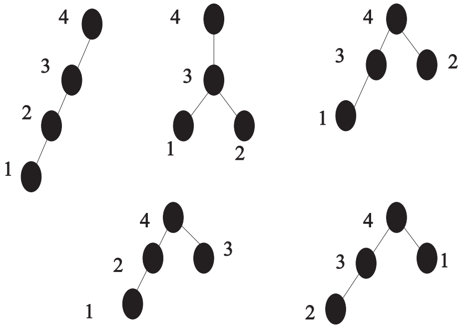

Figure 7.2
The five rooted 1-2 trees on vertex set [4].

See Figure 7.2 for an illustration. Let $T(x) = \sum_{n\geq 0} T_n \frac{x^n}{n!}$ be the exponential generating function of the sequence of the numbers T_n. Find the exponential growth rate of the sequence $T_n/n!$.

7. Recall that the Stirling number of the second kind, $S(n,k)$, denotes the number of partitions of $[n]$ into k blocks. Let $F_k(x) = \sum_{n\geq k} S(n,k)x^k$. In other words, $F_k(x)$ is the generating function of the Stirling numbers $S(n,k)$ with k fixed.
   (a) Prove that, if $k \geq 2$, then
   $$F_k(x) = \frac{x}{1-kx}F_{k-1}(x).$$
   (b) Compute the exponential growth rate of the sequence $\{S(n,k)\}_{n\geq k}$.
8. Let t_n be the number of rooted plane trees on n unlabeled vertices in which each vertex has at most two children, and let $t_0 = 0$. See Figure 7.3 for an illustration. Find the exponential growth rate of

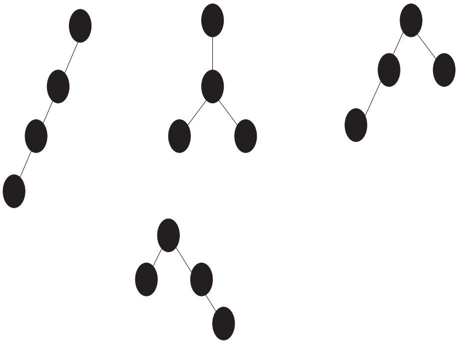

Figure 7.3
The four plane 1-2 trees on four unlabeled vertices.

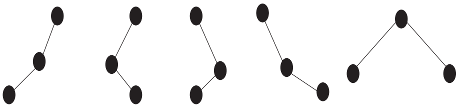

Figure 7.4
The five binary plane trees on three vertices.

the sequence of the numbers t_n. (Note that we have computed the numbers t_n in Exercise 32 of Chapter 5 using the Lagrange Inversion Formula, but do not use that result here; use generating functions instead and singularity analysis instead.)

9. Let T_n be the number of *binary* plane trees on n unlabeled vertices. In such trees, each vertex has at most two children, and a child of a vertex can be a left child or a right child, even if it is an only child. Figure 7.4 shows the five binary plane trees on three vertices. Set $T_0 = 1$. Find the exponential growth rate of the sequence of the numbers T_n. Where have you seen these numbers?
10. Let b_n be the number of rooted plane trees on n unlabeled vertices in which any non-leaf vertex can have any number of children. These trees were formally defined in Definition 5.26. Set $b_0 = 0$. Find the exponential growth rate of the sequence of the numbers b_n.
11. Let m_n be the number of ways to have a group of n people split into nonempty subsets, to have each subset sit down around a circular table, then to arrange the tables in a circle. Find the exponential growth rate of the sequence $m_n/n!$. Two arrangements are considered identical if each person has the same left neighbor in them, and each table has the same left neighbor in them.
12. Let t_n be the number of undirected graphs on vertex set $[n]$ in which each connected component is a cycle of length at least three. Find the exponential generating function for the numbers t_n, then find the exponential growth rate of the sequence $t_n/n!$.
13. Let p_n be the number of undirected graphs on vertex set $[n]$ in which each connected component is a path. Find the exponential growth rate of the sequence $p_n/n!$. (Components that consist of a single vertex are allowed.)
14. A baseball coach asks his n players to split into a few subsets that consist of at least one person each, then he asks that the set of groups stand around a cycle. (There is no structure imposed on each set of players.) Let c_n be the number of ways in which this can happen. Find the exponential order of the sequence $c_n/n!$. Two arrangements are considered the same if the partition of the players is the same in both, and each set of players has the same set as its left neighbor in both.

15. A football coach partitions the set of his n players into blocks of players so that each block consists of an odd number of players. Then he asks each block to form a line. Let L_n be the number of ways in which this can happen. Find the exponential growth rate of the sequence $L_n/n!$. (There is no structure imposed on the set of blocks.)

16. Let us modify the situation described in the previous exercise by imposing a linear order on the blocks of players (so that there is now a first block, a second block, and so on). In other words, we arrange the set of blocks in a line. Let w_n be the number of ways that the combined task can be carried out. Find the exponential growth rate of the sequence $w_n/n!$.

17. A group of football players stand in a line. The coach walks by the line and splits the line into a few segments. (The coach has the option to leave the entire line as one segment.) Then she asks each segment to choose three captains. Let b_n be the number of ways in which this can happen. Find the exponential growth rate of the sequence of the numbers b_n.

18. Let $f(n)$ be the number of words of n letters over the alphabet $\{A, B\}$ that do not contain a subword ABA in consecutive positions. Find the exponential growth rate of the sequence $f(n)$.

19. Consider the undirected cycle C_n on vertex set $[n]$, and let us call a subset S of its vertices a *maximal independent set* if there is no edge between any two vertices in S, but adding any new vertex to S would destroy that property. For $n \geq 3$, let P_n denote the number of maximal independent sets of C_n. Find the exponential growth rate of the sequence P_n.

20. Let A_n be the number of *all* independent sets of any size in the path on vertex set $[n]$. Find the exponential growth rate of the sequence A_n. An independent set is a set of vertices so that there are no edges between vertices of the set.

21. Let $n \geq 3$, and let B_n be the number of all independent sets of any size in the circle C_n on vertex set n. Find the exponential growth rate of the numbers B_n.

22. Generalizing Example 7.12, let $a_{n,k}$ be the number of outcome sequences of n coin flips that do not contain k consecutive heads. Find the closed form of the exponential generating function $A_k(x) = \sum_{n\geq 0} a_{n,k}x^n$, and explain how its singularity of smallest modulus will change if k changes.

23. A *decreasing plane 1-2 tree* on vertex set $[n]$ is a plane tree whose vertices are bijectively labeled by the elements of $[n]$ so that each vertex has a smaller label than its parent. Note that the only difference between these trees and the decreasing binary trees that were