introduced in Definition 5.31 is that, in decreasing plane 1-2 trees, if a child is the only child of its parent, then it does not have a "direction," that is, it is not a left child or a right child. Set $h_0 = 0$, and let h_n be the number of decreasing plane 1-2 trees for $n \geq 1$. Find the exponential growth rate of the sequence $h_n/n!$. You may use a software package to solve any differential equations that you need for your solution.

7.7 Solutions to exercises

1. This is because
$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)},$$
so those two rational functions only differ in a polynomial, that is, their coefficients of x^n only differ for a finite number of exponents n.

2. Let us assume that z is not a positive real number, and $z^k + z^\ell = 1$. Then, by the triangle inequality, we have $|z|^k + |z|^\ell > 1$. Consider the increasing function $f(t) = t^k + t^{ell}$ defined on the positive reals. As $f(|z|) > 1$, and $f(0) = 0$, it follows from the continuity of f that there exists a $t \in (0, |z|)$ for which $f(t) = 1$.

   The triangle inequality is strict, unless z^k and z^ℓ are vectors that point into the same direction, that is, unless $z^{k-\ell}$ is a positive real number. However, as $1 = z^k + z^\ell = (1 + z^{k-\ell})z^\ell$, that would imply that z^ℓ is also a positive real number. That means that both ℓ and $k - \ell$ are multiples of $2\pi/Arg(z)$, a contradiction.

3. It suffices to prove that, for all positive real numbers x, the inequality $x^4 + x > x^3 + x^2$ holds. That inequality holds since it is equivalent to
$$\begin{aligned} x^4 + x - x^3 - x^2 &> 0 \\ (x+1)x(x-1)^2 &> 0, \end{aligned}$$
which obviously holds for all positive real numbers x. Now we know from the previous exercise that the root of smallest modulus of both $Q_a(x)$ and $Q_b(x)$ is a positive real number. Let r_0 be that root of $Q_a(x)$, and let r_1 be that root of $Q_b(x)$. Then $r_0 + r_0^4 = 1$, and it follows from the inequality that we just proved that $r_1^2 + r_1^3 < 1$. As $f(x) = x^2 + x^3$ is obviously an increasing function for positive x, the statement is proved.

4. We use the compositional formula of exponential generating functions, with $A = e^x - 1$ and $B(x) = 1/(1-x)$. Indeed, we split $[n]$ into k blocks, and then there is no task to carry out on these blocks other than making sure that they are nonempty. Then, we order the set of these blocks linearly, and we map the elements of the first block into 1, the elements of the second block into 2, and so on. There are $k!$ ways to linearly order k blocks, showing that $b_k = k!$, and so $B(x) = \sum_{k\geq 0} k! \cdot \frac{x^k}{k!} = 1/(1-x)$.

   Therefore, the compositional formula yields

$$H(x) = B(A(x)) = \frac{1}{1-A(x)} = \frac{1}{1-(e^x-1)} = \frac{1}{2-e^x}.$$

   The function e^x is an entire function; therefore, the only way in which $H(x)$ can have a singularity is by $e^x = 2$, There are infinitely many complex numbers x that satisfy $e^x = 2$, but the one with the smallest modulus is the real number $\ln 2$. Therefore, the exponential order of the numbers h_n is $1/\ln 2 \approx 1.4427$. Note that the numbers h_n are called the *surjection numbers.*

5. As the number of bijections on $[n]$ is $n!$, the ratio $h_n/n!$ is the ratio of all surjections defined on $[n]$ (and mapping to any $[k]$) and the bijections on $[n]$. The result of the previous exercise says that, for large n, there will be about 1.4427^n times as many surjections as bijections.

6. We have seen in Section 5.5.2.2 that $T(x) = \sum_{n\geq 0} T_n \frac{x^n}{n!} = \sec x + \tan x$. As both $\sec x$ and $\tan x$ have their singularities of smallest modulus at distance $\pi/2$ from 0, it follows that the exponential growth rate of the sequence $T_n/n!$ is $2/\pi$.

7. (a) Consider the recurrence relation proved in Theorem 2.10. Multiply both sides by x^n, then sum over all $n \geq k$ to get $F_k(x) = xF_{k-1}(x) + kxF_k(x)$, which is equivalent to our claim.

   (b) Note that $F_1(x) = x/(1-x)$. Then the recurrence we proved in part (a) and induction on k leads to the explicit formula

$$F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

   So the singularity of smallest modulus of $F_k(x)$ is at $x_0 = 1/k$, proving that the exponential growth rate of the sequence of its coefficients is k.

8. Let $t(x)$ be the generating function of the sequence. Removing the root of such a tree, we get either the empty set, or one such tree, or the ordered pair of two such trees. This leads to the functional equation

$$t(x) = x + xt(x) + xt(x)^2.$$

This is a quadratic equation for $t(x)$, whose solutions are

$$t_{1,2}(x) = \frac{1 - x \pm \sqrt{(x-1)^2 - 4x^2}}{2x}.$$

We know that $t(0) = 0$, so we need the solution that satisfies that requirement, that is, the one with the *negative* sign before the square root.

This yields

$$t(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

So the singularity of $t(x)$ that is of smallest modulus is at $1/3$, hence the exponential growth rate of the sequence t_n is 3. It may seem at first that $x = 0$ is a singularity of t. However, we have defined $t(0) = 0$, and that definition keeps t analytic at 0. That is the only definition of $t(0)$ that achieves that, since $\lim_{x\to 0} t(x) = 0$. Note that the numbers $t_n = M_{n-1}$, where the numbers M_n are the *Motzkin numbers* that we saw in the solution of Exercise 23 of Chapter 3.

A word on terminology: We say that $x = 0$ is a *removable* singularity of the function $s(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$ since s becomes analytic at 0 if we define its value at 0 properly. For general s, this is possible if and only if s is analytic in a neighborhood of 0 except at 0 itself, and $\lim_{x\to 0} s(x) = L$ exists and is finite. In that case, setting $s(0) = L$ will make s analytic at 0. For instance, $s(x) = -\ln(1-x)/x$ has a removable singularity at 0, since $\lim_{x\to 0} s(x) = 1$. So we can set $s(0) = 1$.

9. Removing the root of such a tree (which we can do if the tree is not empty), we get an ordered pair of such trees, some or both of which may be empty. This leads to the functional equation

$$T(x) - 1 = xT(x)^2.$$

Solving this quadratic equation for $T(x)$, we get

$$T(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

We know that $T(0) = 1$, so we need the solution that satisfies that requirement, that is, the one with the *negative* sign before the square root. Therefore, $T(x) = \frac{1-\sqrt{1-4x}}{2x}$, and the singularity of the smallest modulus of $T(x)$ is $1/4$, hence the exponential growth rate of the sequence T_n is 4. Note that $x = 0$ is not a singularity, since we have defined $T(0) = 1$, which equals $\lim_{x\to 0} T(x)$. Just as in the previous exercise, the reader should explain why $x = 0$ is not a singular point of T. In other words, 0 is a removable singularity of $\frac{1-\sqrt{1-4x}}{2x}$ (and not a singularity of $T(x)$).

It follows from the closed form of the generating function $T(x)$ that the numbers T_n are the Catalan numbers.

10. We proved in Section 5.5.2.1 that

$$B(x) = \frac{1-\sqrt{1-4x}}{2},$$

so $B(x)$ has only one singularity, at $x = 1/4$. Therefore, the exponential order of the sequence b_n is 4. Note that $b_n = c_{n-1}$, the $(n-1)$st Catalan number.

11. We use the compositional formula of exponential generating functions. The inside function is $\ln(1/(1-x))$ and the outside function is $1 + \ln(1/(1-x))$. Therefore, we get

$$M(x) = 1 + \ln(1 - \ln(1/(1-x)).$$

Clearly, $M(x)$ has a singularity at $x = 1$, and also when $1 = \ln(1 - \ln(1/(1-x))$, which occurs if $1 = \ln(1/(1-x))$, that is, when $1/(1-x) = e$, or $x = 1 - e^{-1}$. This second singularity is closer to 0 than the first, so the exponential growth rate of the sequence of the m_n is $1/(1-e^{-1}) = 1 + \frac{1}{e-1} \approx 1.582$.

12. We will use the exponential formula. There are $(k-1)!$ ways to put a *directed* cycle on k vertices, so there are $(k-1)!/2$ ways to put an undirected cycle on them, for $k \geq 3$. Therefore,

$$\begin{aligned} A(x) &= \sum_{k\geq 3} \frac{(k-1)!}{2} \cdot \frac{x^k}{k!} \\ &= \sum_{k\geq 3} \frac{x^k}{2k} \\ &= \frac{1}{2}\ln\left(\frac{1}{1-x}\right) - \frac{x}{2} - \frac{x^2}{4}. \end{aligned}$$

Therefore, the exponential formula implies that

$$T(x) = \sum_{n\geq 0} t_n \frac{x^n}{n!} = \exp(A(x)) = \frac{e^{-\frac{x}{2}-\frac{x^2}{4}}}{\sqrt{1-x}}.$$

So $T(x)$ has only one singularity, at $x = 1$, and therefore the exponential growth rate of the sequence $t_n/n!$ is 1.

13. This is similar to the previous exercise. For $n \geq 2$, there are $n!/2$ ways to put an undirected path on n vertices, whereas there is one way to put a path on one vertex. This yields

$$A(x) = x + \frac{1}{2}\sum_{n\geq 2}\frac{n!}{2}\frac{x^n}{n!} = x + \frac{1}{2}\frac{1}{1-x} - \frac{x}{2} - \frac{1}{2} = \frac{1}{2(1-x)} + \frac{x-1}{2}.$$

As $A(x)$ has only one singular point, at $x = 1$, so does $P(x) = \sum_{n\geq 0} p_n x^n = \exp(A(x))$, so the sequence of the numbers $p_n/n!$ has exponential growth rate 1.

14. We apply the compositional formula of exponential generating functions. The inside function is $A(x) = \sum_{n\geq 1}\frac{x^n}{n!} = e^x - 1$, since there is no structure on each set of players. As the sets of players are arranged in a cycle, the outside function is $B(x) = 1 + \sum_{n\geq 1} x^n/n = 1 + \ln(1/(1-x))$. Therefore, the compositional formula implies

$$\begin{aligned} H(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!} &= B(A(x)) = 1 + \ln\left(\frac{1}{1-(e^x-1)}\right) \\ &= 1 + \ln\left(\frac{1}{2-e^x}\right). \end{aligned}$$

So the singularity of smallest modulus is at $r_1 = \ln 2$; therefore, the exponential growth rate of the coefficients $h_n/n!$ is $1/\ln 2 \approx 1.4427$.

15. As there is no structure imposed on the set of blocks, we use the exponential formula. We have $A(x) = \sum_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}(2n+1)! = x/(1-x^2)$. Therefore, the exponential formula implies

$$L(x) = \sum_{n\geq 0} L_n \frac{x^n}{n!} = \exp(A(x)) = \exp\left(\frac{x}{1-x^2}\right),$$

so $L(x)$ has two singularities, both of modulus 1. Hence the exponential growth rate of the coefficients $L_n/n!$ is 1.

16. Now we have to use the compositional formula for exponential generating functions, with $A(x)$ being the same as in the previous exercise, but $B(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = 1/(1-x)$. We obtain

$$W(x) = \sum_{n\geq 0} w_n \frac{x^n}{n!} = \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2}.$$

The singularity of smallest modulus is $r_0 = (\sqrt{5}-1)/2$, so we obtain the exponential growth rate of the sequence $w_n/n!$ as $1/r_0 = (\sqrt{5}+1)/2 \approx 1.618$.

17. As the players are standing in a line, we need to use *ordinary* generating functions, namely, Theorem 3.25. The inside function is

$$\begin{aligned} A(x) &= \sum_{k\geq 3} \binom{k}{3} x^k \\ &= x^3 \sum_{k\geq 3} \binom{k}{3} x^{k-3} \\ &= \frac{x^3}{6} \cdot \left(\frac{1}{1-x}\right)''' \\ &= \frac{x^3}{(1-x)^4}. \end{aligned}$$

Hence Theorem 3.25 yields

$$B(x) = \sum_{n\geq 0} b_n x^n = \frac{1}{1-A(x)} = \frac{(1-x)^4}{1-4x+6x^2-5x^3+x^4}.$$

So $B(x)$ is a rational function. Its singularity of smallest modulus is at $r_1 = 0.4503$, so the exponential growth rate of the coefficients b_n is $1/r_1 \approx 2.2207$.

18. We will make repeated use of Theorem 3.25. Let $f(0) = 1$, and let $F(x) = \sum_{n\geq 0} f(n)x^n$. If an allowed word starts with a letter B, then that letter B can be followed by any allowed word. Such words are counted by the generating function $xF(x)$. Otherwise, let us look for the first letter B in our word w. This letter B is immediately preceded by a string of letters A, so it cannot be immediately followed by a letter A. Therefore, after the first letter B, the word w either ends or continues with another letter B, and then with any allowed word. In the first case, w is counted by the generating function $\sum_{n\geq 2} x^n = x^2/(1-x)$ since w is just a nonempty sequence of letters A, and then a single B. In the second case, w is counted by the generating function $\frac{x^2}{1-x} \cdot x \cdot F(x)$. Finally, w can be the empty word, which has generating function 1, or the word that consists of letters A only, which has generating function $x/(1-x)$.

    This leads to the functional equation

$$F(x) = xF(x) + \frac{x^2}{1-x} + \frac{x^2}{1-x} \cdot x \cdot F(x) + 1 + \frac{x}{1-x}.$$

Solving this equation, we get

$$F(x) = \frac{1+x^2}{1-2x+x^2-x^3}.$$

The singularity of $F(x)$ that has smallest modulus is $r_1 \approx 0.5698$, so the exponential growth rate of the sequence $f(n)$ is $1/r_1 \approx 1.755$.

19. We claim that $P(n) = P(n-3) + P(n-2)$ if $n \geq 6$, with $P(3) = 3$, $P(4) = 2$, and $P(5) = 5$. In order to prove this claim, let $n \geq 6$. Let i be the smallest vertex in S, and let j be the second smallest vertex in S. Then either $j - i = 2$ or $j - i = 3$. In either case, contract the entire path $i, i+1, \cdots j$ into one vertex, call that vertex i, and decrease the label of all vertices larger than j by $j - i$. Form a new set S' from the new vertex i and the images of the other vertices that were in S. Then S' is a maximal independent set in the obtained C_{n-2} or C_{n-3}. Furthermore, this construction is bijective, since, starting with a maximal independent set T of a C_{n-2} or C_{n-3}, we can find its preimage by finding the smallest i vertex in T, and then replacing it by a path of three or four vertices, the two endpoints of which are in the preimage of T.

    In order to find the generating function $P(x) = \sum_{n\geq 0} P(n)x^n$, we may set $P(2) = 2$, $P(1) = 0$, and $P(0) = 3$ to keep the recurrence true for all values of $n \geq 3$. Set $P(x) = \sum_{n\geq 0} P(n)x^n$. Then the recurrence leads to the functional equation

$$P(x) - 2x^2 - 3 = x^3 P(x) + x^2(P(x) - 3),$$

$$P(x) = \frac{-x^2 + 3}{1 - x^2 - x^3}.$$

    Finding the singularity of smallest modulus of this rational function, we conclude that the exponential growth rate of the numbers $P(n)$ is 1.3247. The numbers $P(n)$ are called the *Perrin numbers*.

20. Let $A(x) = \sum_{n\geq 0} A_n x^n$. Clearly, we have $A_0 = 1$, $A_1 = 2$, (note that the empty set is independent as well), and $A_n = A_{n-1} + A_{n-2}$ for $n \geq 2$, since there are A_{n-1} such sets not containing n, and A_{n-2} such sets containing n. This leads to the functional equation

$$A(x) - 2x - 1 = x(A(x) - 1) + x^2 A(x),$$

$$A(x) = \frac{x + 1}{1 - x - x^2},$$

    hence the exponential growth rate of the sequence A_n is $(\sqrt{5}+1)/2$. Note that the number A_n is the $(n+1)$st Fibonacci number.

21. Independent sets S of the path on vertex set $[n]$ will be independent sets in C_n if and only if they do not contain *both* 1 and n. If S does contain both 1 and n, then the rest of S is an independent set in the path from vertex 3 to vertex $n - 2$, so in a labeled path of $n - 4$ vertices, if $n \geq 5$. So for $n \geq 5$, we have $B_n = A_n - A_{n-4}$. Setting

$B_0 = B_1 = B_2 = 0$, and noting that $B_3 = 4$ and $B_4 = 7$, we have (with $B(x) = \sum_{n \geq 0} B_n x^n$),

$$B(x) = A(x)\left(1 - x^4\right) - 1 - 2x - 3x^2 - x^3,$$

so $B(x)$ has the same set of singularities as $A(x)$, hence the sequence B_n has the same exponential growth rate as the sequence A_n, and we have computed that one in the solution of the previous exercise.

Note that the numbers B_n are called the *Lucas numbers* and they satisfy the recurrence relation $B_n + B_{n+1} = B_{n+2}$ for $n \geq 3$, with the initial conditions mentioned above.

22. Analogously to the solution of Example 7.12, we get the functional equation

$$A_k(x) = A_k(x)x\left(1 + x + \cdots + x^{k-1}\right) + \left(1 + x + \cdots + x^{k-1}\right).$$

This yields the closed formula

$$A_k(x) = \frac{\left(1 + x + \cdots + x^{k-1}\right)}{1 - x - x^2 - \cdots - x^k}.$$

That is, the singularity of smallest modulus of $A_k(x)$ is the root of $f_k(x) = x + \cdots + x^{k-1} - 1$ that is closest to 0. It is easy to prove, and you are asked to do so in Supplementary Exercise 4 in more general circumstances, that $f_k(x)$ has a unique such root, and that root is a positive real number r_k. It follows that $r_{k+1} < r_k$, since f_{k+1} has an additional summand over f_k, namely, x^k. Therefore, the exponential growth rate of the sequence $a_{n,k+1}$ is larger than that of the sequence $a_{n,k}$. This is what we expected, since it is easier to avoid a sequence of $k + 1$ consecutive heads than it is to avoid k consecutive heads.

23. The procedure is quite similar to the one that we applied in Section 5.5.2.2 to find the number of *nonplane* decreasing 1-2 trees on vertex set $[n]$. Let $H(x) = \sum_{n \geq 0} h_n x^n / n!$. If $n \geq 2$, then removing the root of a plane decreasing 1-2 tree in vertex set $[n]$, we either get an *ordered* sequence of two trees with combined vertex set $[n - 1]$, or just one tree with vertex set $[n - 1]$. If $n = 1$, then we get the empty set. This leads to the differential equation

$$H'(x) = H(x)^2 + H(x) + 1.$$

Using the initial condition $H(0) = 0$, we can solve this differential equation with our favorite software package. After simplification, we get the answer

$$H(x) = -\frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \tan\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}x\right).$$