8

Symmetric structures

8.1 Designs

In Chapter 6, we considered families of sets from the perspective of extremal combinatorics. Keeping the tradition of that field, we called these families of sets hypergraphs, alluding to the fact that they were generalizations of graphs.

In the present chapter, we will consider families of sets again, but we will be interested in their *symmetries* instead. Following the tradition of this line of work, in this chapter, we will call our families of sets *designs*.

That is, a *block design* B is a finite set of elements V, called the *vertices* of the design, together with a collection E of nonempty subsets of V, called the *blocks*. We will keep a few words from the terminology of the previous chapter. In particular, we call a block design B *uniform* or k-uniform if each block of B contains the same number k of vertices.

In real life, we often need more symmetry than that in a block design. For instance, if a soccer coach wants to try out his v new offensive players in various k-striker formations, he may want to give equal amounts of playing time to each of his players. In that case, he would need a *regular*, or r-*regular* block design, that is, a design in which each vertex appears in the same number r of blocks. It goes without saying that the vertices are the players and the blocks are the formations.

The coach may be even more sensitive to fairness, and may want to make sure that any *pair* of strikers plays together in the same number λ of trials. If that happens, then we say that the design the coach uses is *balanced*.

If a design is balanced, uniform, and regular, then it is called a *block design*. Finally, if $k < v$, so the blocks of a block design do not contain all vertices, then that design is called a *balanced incomplete block design*, or BIBD.

This can certainly all be achieved if the coach tries out all $\binom{v}{k}$ possible formations, but that may not end before the season is over. It is therefore interesting to look for designs with a smaller number of blocks that still satisfy all these criteria.

In what follows, we will take a look at designs with one or more of the symmetric properties described above. Let us set the following notation for the rest of this section.

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In a design $\mathcal{F}$,

1. the number of vertices will be denoted by v,
2. the number of blocks will be denoted by b,
3. if $\mathcal{F}$ is uniform, then the number of vertices of any (equivalently, each) of its blocks will be denoted by k,
4. if $\mathcal{F}$ is regular, then the number of blocks in which any (equivalently, each) of its vertices appears will be denoted by r, and
5. if $\mathcal{F}$ is balanced, then the number of blocks in which any (equivalently, each) pair of its vertices appear together will be denoted by λ.

We often mention the parameters of a BIBD by saying that it is a (b, v, r, k, λ)-BIBD.

For the rest of this chapter, we will assume that our designs have more than one block, otherwise a design on $[n]$ could consist of one block that is $[n]$ itself. It turns out that this design would constitute a unique counterexample for many theorems. Recall from the previous chapter that we do *not* allow designs to have repeated blocks.

Example 8.1 *The design $\mathcal{F}$ whose blocks are the k-element subsets of $[n]$ is regular, uniform, and balanced.*

Solution: Each vertex occurs in $r = \binom{n-1}{k-1}$ blocks, so $\mathcal{F}$ is regular. Each block contains k vertices, so $\mathcal{F}$ is uniform. Finally, any two vertices appear together in $\lambda = \binom{n-2}{k-2}$ blocks, so $\mathcal{F}$ is balanced. $\diamondsuit$

It is not surprising that this many symmetries seriously restrict the structure of a design. The following two propositions start exploring the restrictions on the five parameters themselves.

Proposition 8.2 *Let $\mathcal{F}$ be a uniform regular design. Then*

$$bk = vr.$$

Proof: Let us count all pairs (a, B) so that a is a vertex of the block B. If we count by the blocks, then this number is bk, since each of the b blocks contains k vertices. If we count by the vertices, then this number is vr, since each of the v vertices is contained in r blocks. $\diamondsuit$

We can verify Proposition 8.2 for the design $\mathcal{F}$ of Example 8.1. In that example, $b = \binom{n}{k}$, $v = n$, and the other parameters are computed in the proof of that example. Then we get the equation

$$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1},$$

which is indeed correct.

Proposition 8.3 *Let $\mathcal{F}$ be a* balanced *uniform regular design. Then*

$$\lambda(v-1) = r(k-1).$$

Proof: Let a be any fixed vertex of $\mathcal{F}$. Let us count all pairs (B, c), where a and c are two vertices of the block B. If we count by the vertices c, then, for each of the $v-1$ choices for c, there are λ blocks containing both a and c, which explains the left-hand side. If we count by the blocks B, then each of the r blocks containing a contains $k-1$ other vertices that can play the role of c. This explains the right-hand side. $\diamondsuit$

Verifying this proposition again for $\mathcal{F}$ of Example 8.1, we get the equality

$$\binom{n-2}{k-2}(n-1) = \binom{n-1}{k-1}(k-1).$$

It is important to point out that Propositions 8.2 and 8.3 only establish *necessary* conditions on the existence of certain designs, not *sufficient* conditions. For instance, there is no balanced uniform regular design with $v = 16$, $b = 8$, $k = 6$, $r = 3$, and $\lambda = 1$, even if these numbers satisfy the equalities of the mentioned two propositions. Theorem 8.12 will show why there is no such design.

At this point, the reader may hope for a theorem that establishes a sufficient and necessary condition for the existence of a design with parameters v, b, r, k, and λ. Unfortunately, no such theorem is known. In fact, there are quite a few 5-tuples of parameters (v, b, r, k, λ) when it is not known whether a design with those parameters (or, for shortness, a (v, b, r, k, λ)-design) exists. (The simple fact that the parameters r, k, and λ are present assumes that we are considering balanced uniform regular designs.) in a bit more detail in the next section. Until then, we challenge the reader to decide whether a $(7, 7, 3, 3, 1)$-design exists. In the language of our original example, We will discuss these difficult problems we are asking if a soccer coach with seven strikers can try out seven attacking formations so that each formation consists of three strikers, each striker gets three chances to play, and any pair of players gets to play together exactly once.

We have considered three different properties of designs so far, that is, we considered balanced, uniform, and regular designs. These are all strong properties, which can seriously restrict the structure of a design. Exercise 1 shows that, if a design is balanced and uniform, then it is also regular. After learning this fact, the reader might suspect that a design that is balanced and regular is necessarily uniform (since being regular means roughly the same thing for the vertices as being uniform means for the blocks). This conjecture is false, however, as the following example shows.

Example 8.4 *Let $\mathcal{F}$ be the design on $[n]$ whose blocks are the 2-element and 3-element subsets of $[n]$. Then $\mathcal{F}$ is regular (with $r = n-1+\binom{n-1}{2} = \binom{n}{2}$), and balanced (with $\lambda = 1+n-2 = n-1$), but not uniform.*

At this point, the reader may wonder what causes this subtle difference between uniformity and regularity, which seem to be duals of each other. The answer lies in the nature of the *balanced* property. That property ensures that any two vertices appear together in the same number of blocks. However, we have not defined the dual of this property yet, that is, we have not discussed designs in which the intersection of any pair of blocks has the same size. The following definition fills that gap.

Definition 8.5 *A design is called* linked *if any two of its blocks intersect in the same number μ of vertices.*

Example 8.6 *The design $\mathcal{F}$ of Example 8.1 is linked when $k = 1$ (and in that case, $\mu = 0$) and when $k = n-1$ (and in that case, $\mu = n-2$). For other values of k, the mentioned design is not linked.*

We can now state the counterpart of Exercise 1.

Proposition 8.7 *Let $\mathcal{F}$ be a design that is regular and linked. Then $\mathcal{F}$ is uniform.*

Hopefully, the reader is now thinking that maybe we will not need to work out an independent proof of this proposition; maybe we will be able to deduce it from the result of Exercise 1 more or less effortlessly. If that is the case, we have good news for the reader. There exists a general technique that is very useful in translating statements into their *duals*. (We will finally be able to explain precisely what that word means.)

The crucial definition is the following.

Definition 8.8 *Let $\mathcal{F}$ be a design with vertices $a_1, a_2, \cdots, a_v$ and blocks $e_1, e_2, \cdots, e_b$. Then the* incidence matrix *of $\mathcal{F}$ is the $v \times b$ matrix $M = M_{\mathcal{F}}$ for which*

$$M_{i,j} = \begin{cases} 1 \text{ if } a_i \in e_j, \\ \\ 0 \text{ if } a_i \notin e_j. \end{cases}$$

In other words, the *rows* of $M_{\mathcal{F}}$ correspond to the vertices of $\mathcal{F}$, and the columns of $M_{\mathcal{F}}$ correspond to the blocks of $\mathcal{F}$. The intersection of a row and a column is 1 if the vertex corresponding to the row is contained in the block corresponding to the column.

Example 8.9 *If $\mathcal{F}$ is the design whose vertex set is [4] and whose blocks are $e_1 = \{1,2\}$, $e_2 = \{1,3\}$, $e_3 = \{2,4\}$, $e_4 = \{2,3,4\}$, and $e_5 = \{1,4\}$, then*

$$M_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

One great advantage of incidence matrices is that now we can easily define the *dual* of a design.

Definition 8.10 *The* dual $\mathcal{F}^d$ *of a design* $\mathcal{F}$ *is the design whose incidence matrix is the* transpose *of* $M_{\mathcal{F}}$*.*

If we had stated this definition before Chapter 5, the reader might have asked, What do you mean the dual of $\mathcal{F}$ is *the* design with incidence matrix $M_{\mathcal{F}}^T$? What if there are several designs with the same incidence matrix? By now, however, the reader easily dismisses that concern, noting that all designs with the same incidence matrix are *isomorphic*.

We have not formally defined isomorphism of designs yet, but the definition is very similar to that of graph isomorphism. That is, let $\mathcal{F}$ and $\mathcal{G}$ be two designs. If there exists a bijection f from the vertex set of $\mathcal{F}$ onto the vertex set of $\mathcal{G}$ that takes blocks of $\mathcal{F}$ into blocks of $\mathcal{G}$, then we say that f is an isomorphism and that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

Example 8.11 *In order to construct the dual* $\mathcal{F}^d$ *of the design* $\mathcal{F}$ *given in Example 8.9, we first take the transpose of* $M_{\mathcal{F}}$ *and get the matrix*

$$M_{\mathcal{F}}^T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then we read the column j *of this matrix to see which vertices are contained in* e_j*. We get* $e_1 = \{1, 2, 5\}$*,* $e_2 = \{1, 3, 4\}$*,* $e_3 = \{2, 4, 5\}$*, and* $e_4 = \{3, 4, 5\}$*.*

It is straightforward to translate the discussed properties of designs into the language of incidence matrices. For instance, $\mathcal{F}$ is regular if all rows of $M_{\mathcal{F}}$ contain the same number r of 1s; $\mathcal{F}$ is uniform if all columns of $M_{\mathcal{F}}$ contain the same number k of

1s; and so on. Therefore, $\mathcal{F}$ is regular if and only if $\mathcal{F}^d$ is uniform, while $\mathcal{F}$ is balanced if and only if $\mathcal{F}^d$ is linked, and so on. Proving Proposition 8.7 is now a breeze.

Proof: (of Proposition 8.7) If $\mathcal{F}$ is regular and linked, then $\mathcal{F}^d$ is uniform and balanced, and therefore, by Exercise 1, $\mathcal{F}^d$ is regular. Therefore, the dual of $\mathcal{F}^d$, that is, $\mathcal{F}$, is uniform. $\Diamond$

The concept of incidence matrices is useful in proving much more difficult results than Proposition 8.7, as we will see shortly.

We have proved two results (Propositions 8.2 and 8.3) stating equalities involving products of parameters of designs, and Supplementary Exercise 1 provides another such result. What about inequalities, though? Is it true that some parameters are always larger than others? There are, of course, the trivial inequalities $k \leq v$ in each uniform design, and $r \leq b$ in each regular design.

If $\mathcal{F}$ is balanced, then $\lambda \leq r$, since the total number of blocks containing a vertex x is at least as large as the number of blocks in which x occurs together with another vertex y. In fact, $\lambda = r$ is only possible if all vertices occur in all blocks, a case that is really not very exciting.

The following is a much deeper inequality among parameters of a design.

Theorem 8.12 (Fisher's inequality) *Let $\mathcal{F}$ be a balanced uniform design with at least two blocks. Then $v \leq b$.*

Proof: Let M be the incidence matrix of $\mathcal{F}$, and assume that $b < v$. Then M has more rows then columns. Add $v - b$ zero columns to the end of M so that M becomes the $v \times v$ matrix A. Then we see that $MM^T = AA^T$, since the recently added zeros at the end of each row will not change anything. Therefore,

$$\det(MM^T) = \det(AA^T) = \det(A) \cdot \det(A^T) = 0, \tag{8.1}$$

since A has at least one zero column.

On the other hand, the solution of Exercise 3 shows that

$$MM^T = \begin{pmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & \cdots & r \end{pmatrix}.$$

Before the reader asks what r is, let us point out that Exercise 1 shows that a balanced uniform design is *regular*, so r is the number of blocks in which each vertex is contained.

Let us now compute $\det(MM^T)$ by transforming this matrix into triangular form, using elementary row and column operations. First, subtract the first column from all other columns. This does not change the determinant of the matrix, yielding that

$$\det(MM^T) = \det \begin{pmatrix} r & \lambda - r & \lambda - r & \cdots & \lambda - r \\ \lambda & r - \lambda & 0 & \cdots & 0 \\ \lambda & 0 & r - \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda & 0 & 0 & \cdots & r - \lambda \end{pmatrix}.$$

In other words, we will know $\det MM^T$ if we can compute the determinant above. That is encouraging, since the matrix above is almost lower triangular; indeed, its only nonzero entries above the main diagonal are in the first row.

Let us note that the sum of each column, except for the first column, is zero. Therefore, if we add each row to the first, each element of the first row