will become zero except for its first element. As this operation again does not change the determinant of the matrix, we get

$$\begin{aligned}
\det(MM^T) &= \det\begin{pmatrix} r+(v-1)\lambda & 0 & 0 & \cdots & 0 \\ \lambda & r-\lambda & 0 & \cdots & 0 \\ \lambda & 0 & r-\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda & 0 & 0 & \cdots & r-\lambda \end{pmatrix} \\
&= (r+(v-1)\lambda)(r-\lambda)^{v-1} \\
&\neq 0.
\end{aligned}$$

Indeed, the determinant of a triangular matrix is obtained by taking the product of its diagonal entries. In order to see that this product is not zero, note that $r > \lambda$ since $\mathcal{F}$ has at least two blocks. So we get that $\det(MM^T) \neq 0$, contradicting (8.1). That is, the assumption that $b < v$ leads to a contradiction, proving that $v \leq b$. $\Diamond$

An easy consequence of Fisher's inequality is the fact that the dual of a BIBD cannot be a BIBD unless $v = b$, and therefore; $r = k$. Such BIBDs are called *symmetric*, and it turns out that the dual of a symmetric BIBD is indeed a BIBD. You are asked to prove this fact in Exercise 5.

Quick Check

1. Let $\mathcal{D}$ be a BIBD with parameters (b, v, r, k, λ). What relation holds between r and k?
2. Let $\mathcal{D}$ be a BIBD in which $v = 12$, $r = 4$, and $\lambda = 1$. Determine the value of b and k.
3. Does there exist a BIBD with parameters $v = 8$, $b = 70$, and $r = 35$?

8.2 Finite projective planes

In this section, we are going to discuss *finite projective planes.* This name suggests that the objects at hand are geometrical. There is some truth to that suggestion; indeed, as we will see, finite projective planes consist of lines and points. However, in contrast to Euclidean geometry, any two lines will intersect. There will be no parallel lines. On the other hand, we will see that finite projective planes are in fact designs with many of the symmetric properties we studied in the previous section.

Definition 8.13 *A finite projective plane is a collection of a finite set $\mathcal{P}$ of* *points and a finite set $\mathcal{L}$ of lines, where the lines themselves are subsets of $\mathcal{P}$, satisfying the following axioms:*

- *Any two points are in exactly one common line.*
- *Any two lines intersect in exactly one point.*
- *The set $\mathcal{P}$ contains four distinct points, no three of which are on a common line.*

The following classic example is called the *Fano plane.* Soon, we will be able to prove that it is actually the *smallest* finite projective plane.

Example 8.14 *Figure 8.1 shows a finite projective plane in which $\mathcal{P} = [7]$ and the lines are $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 5, 7\}$, $\{2, 6, 7\}$ $\{3, 4, 7\}$, and $\{4, 5, 6\}$.*

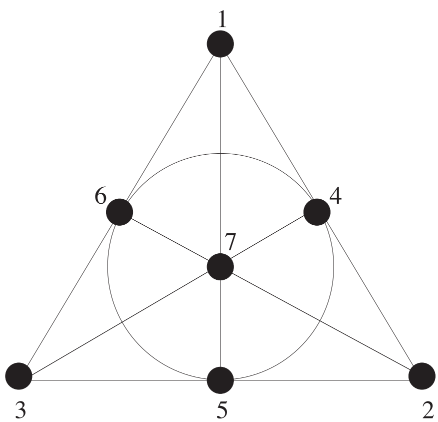

Figure 8.1
The Fano plane.

This example shows that, if we want to represent a finite projective plane by a figure, we can make the task easier if we do not use *straight* lines to represent the lines.

Definition 8.13 shows that a finite projective plane is in fact a *balanced linked design*, in which the points are the vertices, the lines are the blocks, and $\lambda = \mu = 1$. In what follows, we show that this design is uniform and regular as well.

Lemma 8.15 *In a finite projective plane, all points are contained in the same number of lines.*

Proof: Let s and t be two points, and let L be a line that does not contain either one of them. Such an L must exist because of the third axiom in the definition of finite projective planes.

Say L consists of $n+1$ points. Then there is a bijection f between the set of these points and the lines containing s. Indeed, each line S containing s must intersect L in a point a_S. Then the map defined by $f(S) = a_S$ is a bijection with the mentioned properties. Therefore, s is contained in $n+1$ lines. See Figure 8.2 for an illustration.

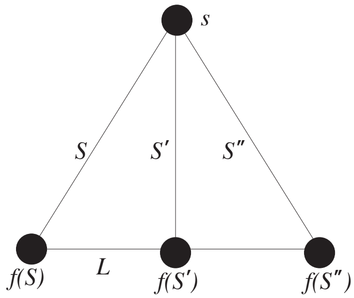

Figure 8.2
The bijection f maps the lines containing s into points of L.

An analogous argument (in which s is replaced by t) shows that t is contained in $n+1$ lines as well. Iterating this argument, we see that any point is contained in the same number $n+1$ of lines as the point s, proving our claim. $\diamondsuit$

Corollary 8.16 *In a finite projective plane, all lines consist of the same number of points. This number $n+1$ is equal to the number of lines containing each point.*

Proof: This follows from the existence of the bijection f just constructed in the proof of Lemma 8.15. $\diamondsuit$

Therefore, each finite projective plane is a regular uniform design, with $k = r = n+1$. By Proposition 8.2, this implies that $v = b$, that is, each finite projective plane has as many points as lines. But *how many*? It is not difficult to answer that question.

Proposition 8.17 *If the lines of a finite projective plane $\mathcal{H}$ consist of $n+1$ points each, then $\mathcal{H}$ has n^2+n+1 points and n^2+n+1 lines.*

Proof: Consider all lines containing point h of $\mathcal{H}$. By Lemma 8.15, there are $n+1$ such lines. By Corollary 8.16, each of these lines contains n points other