2. Necessary condition for existence of a perfect m-error correcting code over a q-element alphabet, consisting of w words of length n each:

$$w \sum_{d=0}^{m} \binom{n}{d} (q-1)^d = q^n.$$

(D) Counting symmetric structures

1. Let G be a finite permutation group acting on a set S, and let $i \in S$. Then

$$\frac{|G|}{|G_i|} = \left| i^G \right|.$$

2. If G acts on the set S, then the number of orbits of G is

$$\frac{1}{|G|} \sum_{g \in G} |F_g|,$$

where F_g is the number of fixed points of $g \in G$.

8.7 Exercises

1. Prove that, if a design is balanced and uniform, then it is also regular.
2. Is there a balanced uniform design in which r divides v?
3. + Let M be the incidence matrix of a balanced uniform design $\mathcal{F}$. Express the matrix product MM^T by the parameters of $\mathcal{F}$, the $v \times v$ identity matrix I_v, and the $v \times v$ matrix J_v whose entries are all equal to 1.
4. Let M be the incidence matrix of a BIBD with parameters (b, v, r, k, λ). Prove that the eigenvalues of MM^T are $r - \lambda$ with multiplicity $v - 1$, and $r + \lambda(v-1) = rk$ with multiplicity 1.
5. + Prove that the dual of a symmetric BIBD is a BIBD.
6. Let $n > 1$. Prove that an $(n^2+n+1, n^2+n+1, n+1, n+1, 1)$-design is a finite projective plane.
7. Prove that the dual design of a finite projective plane is also a finite projective plane.
8. A design is called symmetric if $v = b$. Let $\mathcal{F}$ be a balanced uniform symmetric design, with v even. Prove that $r - \lambda$ has to be a perfect square.

Hint: Look at the determinant $\det(MM^T)$ *of the incidence matrix of this design, which we computed in the proof of Theorem 8.12.*

Note: This is the easy part of the famous Bruck–Ryser theorem. That theorem also says that, if v is odd, then the equation

$$x^2 = (r - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2 \tag{8.7}$$

must have an integer solution (x, y, z), where not all of x, y, and z are zero.

9. Uniform designs with $k = 1$ are called *sets*, while uniform designs with $k = 2$ are called *graphs*. The next category, uniform designs with $k = 3$, are often called *triple systems*, and a balanced triple system with $\lambda = 1$ is called a *Steiner triple system*. For instance, the Fano plane is a Steiner triple system.

   (a) Prove that there is no Steiner triple system in which v is even.

   (b) Prove that there is no Steiner triple system in which $v + 1$ is divisible by 6.

10. Prove Lemma 8.23.

11. Will the error-correcting code constructed from a finite projective plane using the method explained in Example 8.25 ever be a perfect code?

12. Prove that, if G is a group, and H is a subgroup of G, then two cosets of H in G are either disjoint or equal. In other words, prove that the cosets of H partition G. (Note that this implies that $|H|$ divides $|G|$.)

13. Use Theorem 8.36 to prove the *small Fermat theorem*, which states that, if x is a positive integer, and p is prime, then $x^p - x$ is divisible by p.

14. Explain why Theorem 5.7 is a special case of Lemma 8.34.

15. Find the number of ways to color each side of a square red or blue or green if two colorings are considered equivalent if they can be moved into each using rotations.

16. Find the number of ways to color the six edges of a regular tetrahedron (that has four sides, each of which is a regular triangle) using k colors if two colorings are considered identical if they can be moved into each other using rotations.

17. + Find the number of ways to color the 12 edges of a cube using k colors if two colorings are considered identical if they can be moved into each using rotations.