

8.8 Solutions to exercises

1. The proof is similar to that of Proposition 8.3. Fix a vertex a that is in r_a blocks, and count all pairs (B, c) , where a and c are two vertices of the block B . Then, by the same argument as in the mentioned proof, we get that $\lambda(v-1) = r_a(k-1)$, so $r_a = \lambda(v-1)/k$, which is independent from a .
2. No. By the result of the previous exercise, such a design would be regular as well, and $\lambda(v-1) = r(k-1)$ would hold. As r divides v , the number r is relatively prime to $v-1$, so λ would have to be a multiple of r . This is impossible, since $\lambda < r$.
3. The (i, j) entry of the MM^T is the dot product of the i th and j th rows of M . So, if $i \neq j$, this entry is λ , since that is the number of blocks containing both a_i and a_j . If $i = j$, then this entry is simply the number of blocks containing a_i , that is, r . Indeed, the solution of Exercise 1 shows that \mathcal{F} is regular. Therefore, we get

$$MM^T = \lambda J_v + (r - \lambda)I_v. \quad (8.8)$$

4. This follows from (8.8) that we obtained at the end of the solution of the previous exercise. Indeed, J_v has $v-1$ eigenvalues equal to 0, since its rank (number of linearly independent rows) is 1. Its last eigenvalue is λv , since the sum of all the eigenvalues of a matrix equals its trace, which is λv in this case. On the other hand, $(r-\lambda)I_v$ is a constant multiple of the identity matrix, so all its eigenvalues are equal to $r-\lambda$. Finally, if a vector \mathbf{x} is an eigenvector of J_v with eigenvalue a , then \mathbf{x} is also an eigenvector of $\lambda J_v + (r-\lambda)I_v$, with eigenvalue $a+r-\lambda$. Setting $a=0$ ($v-1$ times) and $a=\lambda v$ (once) implies the claim of the exercise.
5. The dual of a regular, uniform design is obviously regular and uniform, even if the design is not symmetric. It remains to prove that, if \mathcal{F} is symmetric, then it is *linked*, because that is the sufficient and necessary condition for \mathcal{F}^d to be balanced. In order to prove that \mathcal{F} is linked, it suffices to prove that $M^T M = M M^T$, where M is the incidence matrix of \mathcal{F} . Indeed, all nondiagonal entries in MM^T are identical.

Note that, in the solution of the previous exercise, we computed the eigenvalues of the incidence matrix of *any* BIBD and found that they were not 0. So, if M is a square matrix, then $\det M \neq 0$, and hence M^{-1} exists. Multiply both sides of (8.8) by M^{-1} from the left, and M by the right, to get

$$M^T M = M^{-1}(\lambda J_v + (r - \lambda)I_v)M = \lambda J_v(r - \lambda)I_v.$$

On the other hand, (8.8) shows that $MM^T = \lambda J_v(r - \lambda)I_v$, proving our claim.

6. Because such a design \mathcal{F} is balanced with $\lambda = 1$, the first axiom is satisfied. If blocks A and B intersected in at least two vertices, say a and b , then a and b would be in at least two common blocks, which is not possible. So $|A \cap B| \leq 1$.

Now let us assume that A and B are disjoint. Let x be a vertex that is neither in A nor in B . Such a vertex exists, since $2(n + 1) < n^2 + n + 1$ since $n > 1$. Because \mathcal{F} is balanced with $\lambda = 1$, there would have to be a block joining x to each of the $2(n + 1)$ vertices of $A \cup B$, and all these $2(n + 1)$ blocks would be distinct (why?). That is a contradiction, since \mathcal{F} has only $n + 1$ lines. So $|A \cap B| = 1$ for any two blocks A and B , and therefore the second axiom is satisfied.

Finally, the third axiom is satisfied since we can pick two blocks, C and D , and remove their common vertex y . We are left with $2n \geq 4$ vertices, n in each block. Choose one pair of vertices from each block, then the obtained four vertices have the property that no three of them are in the same block (otherwise, there would be a pair of vertices that are contained together in at least two blocks). This shows that the third axiom is satisfied as well.

7. The only nontrivial part is that the third axiom will hold. This is equivalent to the fact that, in the *original* finite projective plane \mathcal{H} , there exist four lines, no three of which intersect in one point. We will now prove that this is true.

Let a and b be two points of \mathcal{H} . Let L be the unique line joining them. We know that each of a and b are part of $n + 1$ lines, where $n > 1$. Therefore, we can choose two lines A_1 and A_2 distinct from L that contain a (and not b) and two lines B_1 and B_2 distinct from L that contain b (but not a). We claim that these four lines have the required property that no three of them intersect in one point. Assume not; then, without loss of generality, we can assume that A_1, A_2 , and B_1 intersect in one point. However, A_1 and A_2 already intersect in a , so this intersection point must be a . That implies $a \in B_1$, which is a contradiction.

8. On one hand,

$$\det(MM^T) = \det(M) \det(M^T) = (\det(M))^2,$$

so $\det(MM^T)$ is a perfect square.

On the other hand, we computed in the proof of Theorem 8.12 that

$$\det(MM^T) = (r + (v - 1)\lambda)(r - \lambda)^{v-1}.$$

Note that, since our design \mathcal{F} is balanced and uniform, it is regular, so the use of r is justified. We know from Proposition 8.3 that

$(v - 1)\lambda = r(k - 1)$; therefore, the first factor of the right-hand side of the last displayed equation is equal to $r + r(k - 1) = rk = r^2$. Indeed, since \mathcal{F} is symmetric, $r = k$ by Proposition 8.2. So we get

$$(\det(M))^2 = \det(MM^T) = r^2 \cdot (r - \lambda)^{v-1}.$$

This implies that $(r - \lambda)^{v-1}$ is a perfect square. However, $v - 1$ is odd, so $r - \lambda$ must be a perfect square.

9. (a) Because a Steiner triple system is balanced and uniform, it is regular, so $\lambda(v - 1) = r(k - 1)$ holds. However, in our special case, $k = 3$ and $\lambda = 1$, leading to $v = 2r + 1$.
- (b) For Steiner triple systems, the identity $bk = vr$ becomes $3b = vr$. Comparing this to the identity $v = 2r + 1$ of part (a), we get

$$r(2r + 1) = 3b.$$

So either r is divisible by 3, and then $2r + 1 = v$ is of the form $6t + 1$, or $2r + 1 = v$ itself is divisible by 3, in which case, being odd, v must be of the form $6t + 3$. Note that it was proved by Kirkman in 1847 that, if v is of one of these two forms, then a Steiner triple system on v vertices does exist.

10. We can get from x to y by first changing $d(x, z)$ digits to turn x to z , then by changing $d(z, y)$ digits to turn z to y . During this procedure, at most $d(x, z) + d(z, y)$ digits change, so the number of different digits of x and y is at most that.
11. No. For that code to be perfect, we would need $v(1 + v) = 2^v$. That is impossible, since one of v and $1 + v$ must be odd (and at least 7), and a power of 2 cannot have a proper odd divisor.
12. Let aH and bH be two cosets of H in G . Let us assume first that $a \in bH$. That means that there exists $h \in H$ so that $a = bh$. Then $aH = bhH = bH$. Indeed, $hH = H$, since $hH \subseteq H$ (as H is closed under multiplication) and $|hH| = |H|$.

Now let us assume that $a \notin bH$. We claim that then $aH \cap bH = \emptyset$. Assume not; that is, assume rather that $c \in aH \cap bH$. That means $c = ah_1 = bh_2$, with $h_1, h_2 \in H$. However, that implies $a = bh_2h_1^{-1}$, which contradicts $a \notin bH$.

13. Let us color the sides of a regular p -gon with colors from the set $[x]$ so that at least two colors are used. Let us say that two colors are equivalent if they are rotated images of each other. If all sides were distinguishable, the total number of colorings would be $x^p - x$, since we have to use at least two colors.

There are p possible rotations, $r, r^2, \dots, r^p = id$. The nonidentity rotations will not fix any allowed coloring. Indeed, if r^i fixed a coloring C , then the first side of the polygon would have to have the

same color as its $i + 1$ st side, and that would have to have the same color as its $2i + 1$ st side, and so on. As p is prime, p is relatively prime to i , so that would imply that all sides of the polygon are of the same color, which is a contradiction.

Therefore, Theorem 8.36 yields that the number of nonequivalent colorings is

$$\frac{1}{p}(1 \cdot (x^p - x) + (p - 1) \cdot 0) = \frac{x^p - x}{p}.$$

As this is the number of certain colorings, it must be an integer, proving our claim.

14. Let G be the *symmetric* group acting on the set $[n]$ of vertices of our graph. Then G acts on the set of all *labelings* as well. In the latter action, $|i^G|$ is just the number of all labelings of the graph, and $|Aut(H)|$ is the number of orbits of the action of G .
15. Of the four rotations r, r^2, r^3, id , two, namely, r and r^3 , only preserve the colorings in which all sides are of the same color. The identity preserves all 81 colorings. Finally, r^2 preserves those in which opposite sides are of the same color, which happens in nine cases. By Theorem 8.36, this shows that there are

$$\frac{1}{4}(1 \cdot 81 + 1 \cdot 9 + 2 \cdot 3) = 24.$$

16. The tetrahedron has 12 symmetries that can be achieved by a series of rotations, as we showed in Exercise 36 of [Chapter 4](#). The easiest one is the identity, which will fix all k^6 colorings. Then there are the eight rotations around an axis that is perpendicular to one face F of the tetrahedron and contains the fourth vertex D ; these will only fix the colorings in which the three blocks of F are monochromatic and the three blocks adjacent to D are monochromatic. There are k^2 such colorings. Finally, there are three transformations which interchange two pairs of vertices. Without loss of generality, consider the transformation that interchanges both A and B and C and D . This transformation fixes blocks AB and CD , interchanges blocks BC and AD , and interchanges blocks AC and BD . Therefore, this transformation will preserve colorings in which BC and AD have the same color, and AC and BD have the same color. There are k^4 such colorings.

By Theorem 8.36, this yields that the number of nonequivalent colorings is

$$\frac{1}{12} \cdot (k^6 + 8k^2 + 3k^4).$$

The reader is invited to find the 12 nonequivalent colorings for $k = 2$.