9

Sequences in combinatorics

In this chapter, we will consider combinatorial properties of entire sequences, as opposed to their individual elements.

9.1 Unimodality

In the previous chapter, we encountered the following problem in a different context. A professor is preparing questions for an oral examination of a large number of students. Being a math professor, he puts together a list of n questions. He wants to be fair and ask each student the same number k of questions. He does not want to ask the same set of questions twice. What should k be in order to maximize the number of students the professor can test with these conditions?

As $[n]$ has $\binom{n}{k}$ subsets of k elements, the professor needs to find the value of k for which $\binom{n}{k}$ is maximal. A look at Figure 9.1 (which you may remember from Chapter 1), in which the entries in the nth row are the values of $\binom{n}{k}$ for fixed n, suggests that the numbers $\binom{n}{0}, \binom{n}{1}, \cdots$ increase steadily until $k = \lfloor n/2 \rfloor$, and then they decrease steadily. This pattern is an important property of sequences that has its own name.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Figure 9.1
The values of $\binom{n}{k}$. Note that for fixed n, the values $\binom{n}{0}, \binom{n}{1}, \cdots$ form the nth row of the Pascal triangle.

Definition 9.1 *The sequence $a_0, a_1, \cdots, a_n$ of nonnegative real numbers is called* unimodal *if there exists an index m so that $a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_{n-1} \geq a_n$.*

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In other words, a sequence is unimodal if its elements increase steadily, then decrease steadily.

It turns out that our observation about the unimodality of the sequence $\binom{n}{k}_{0\leq k\leq n}$ always holds true.

Proposition 9.2 *For any fixed positive integer n, the sequence $\binom{n}{k}_{0\leq k\leq n}$ is unimodal.*

There are many proofs of this well-known fact. We start with the simplest one, which is computational.

Proof: Let $k \leq \lfloor (n-1)/2 \rfloor$. Then we claim that $\binom{n}{k} \leq \binom{n}{k+1}$, so our sequence is (weakly) increasing. Indeed, we have

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{(n-k)!k!}{(n-k-1)!(k+1)!} = \frac{n-k}{k+1} \geq 1. \tag{9.1}$$

Furthermore, the computation above shows that, if $k > \lfloor (n-1)/2 \rfloor$, then $\binom{n}{k} > \binom{n}{k+1}$, so our sequence is decreasing. This proves our claim. ♢

Thus, the professor can use his questions most efficiently if he sets $k = \lfloor n/2 \rfloor$.

Note that our sequence $\binom{n}{k}_{0\leq k\leq n}$ is not simply unimodal; it is also symmetric, since $\binom{n}{k} = \binom{n}{n-k}$. This implies that the maximum of the sequence must be in the middle.

The above proof was simple, but not very enlightening. It strongly depended on the fact that we had a formula for the number of k-element subsets of $[n]$, and a simple one at that. We would like to develop other methods that can be used to prove unimodality results in cases when this is not the case. We will first illustrate our method, called the *reflection principle*, by using it to re-prove Proposition 9.2. As a bonus, we will get a *combinatorial* proof of the fact that $\binom{n}{k} \leq \binom{n}{k+1}$ for $k \leq \lfloor (n-1)/2 \rfloor$.

To that end, let us return to Example 1.26 of Chapter 1, where we implicitly defined a simple bijection between the set of all k-element subsets of $[n]$ and all northeastern lattice paths from $(0,0)$ to $(n-k,k)$. For easy reference, the bijection f works as follows: If the elements of a k-element subset $A \subseteq [n]$ are $a_1, a_2, \cdots, a_k$, then $f(A)$ is the northeastern lattice path whose k north steps are the a_1st, a_2nd, $\ldots$, a_kth steps of the lattice path.

Example 9.3 *Figure 9.2 shows the images of two 4-element subsets of* $[8]$.

Because of this bijection, we can present the rest of our proof in terms of lattice paths instead of k-element subsets of $[n]$. We propose the following strategy. Let L_k be the set of northeastern lattice paths from $(0,0)$ to $(n-k,k)$, and let $k \leq \lfloor (n-1)/2 \rfloor$. We will show that, in that case, there exists an *injection*, that is, a one-to-one map $g : L_k \to L_{k+1}$. This will prove that $|L_k| \leq |L_{k+1}$, which then implies the unimodality of our symmetric sequence $\binom{n}{k}_{0\leq k\leq n}$.

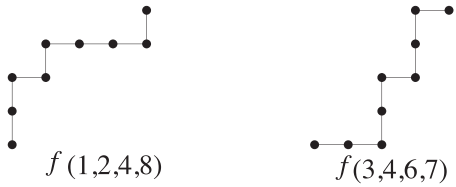

Figure 9.2
The images of $\{1, 2, 4, 8\} \subseteq [8]$ and $\{3, 4, 6, 7\} \subseteq [8]$.

Let $p \in L_k$. Since $k \leq \lfloor n-1/2 \rfloor$, this means that the endpoint $P = (n-k, k)$ of p is below the main diagonal. Let Q be the point $(n - k - 1, k + 1)$, the point where the paths belonging to L_{k+1} end. Let t denote the bisector of the segment PQ. Then t is the line $y = x + (n - 2k) - 1$, and the reader is invited to verify that the two endpoints of p are on *opposite sides* of t. Indeed, the intersection of t and the horizontal axis is the point $(0, n - 2k - 1)$, which is in the positive half of the horizontal axis thanks to our condition on k.

Therefore, p and t intersect. Let X be the last (most northeastern) of their intersection points. Let us *reflect* the partial path XP through t to get a partial path XQ. Now define $g(p)$ as the part of p that goes from $(0, 0)$ to X, followed by the path XQ we just obtained by reflection. This method of proving unimodality statements is called the *reflection principle* and is illustrated with additional examples in Bruce Sagan's survey article [63].

Example 9.4 *Figure 9.3 shows how g maps the lattice path associated with $\{3, 4, 8\} \subset [8]$ into the lattice path associated with $\{3, 4, 6, 7\}$.*

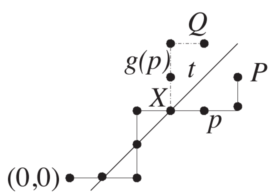

Figure 9.3
The reflection principle at work.

Note that $g(p)$ and t always intersect. The following lemma shows that the reflection principle indeed works.

Lemma 9.5 *For all positive integers $k \leq \lfloor (n-1)/2 \rfloor$, the map $g : L_k \to L_{k+1}$ defined above is an injection.*

Proof: Let $q \in L_{k+1}$, and let P, Q, and t be defined as above. If q and t do not intersect, then q is not in the range of g. Otherwise, let X be the most