northeastern intersection point of q and t. Then the unique preimage of q under g is obtained by reflecting the part of q between X and Q through t, and leaving the rest of q unchanged. ♢

The reflection principle can be applied to more complicated structures as well. The interested reader should look at Chapter 8 of [13], where this technique is applied to decreasing binary trees to prove a unimodality result.

Quick Check

1. Let $\{a_k\}_{0 \leq k \leq n}$ and $\{b_k\}_{0 \leq k \leq n}$ be unimodal sequences. Is it true that $\{a_k + b_k\}_{0 \leq k \leq n}$ is a unimodal sequence?
2. Is it true that all subsequences of a unimodal sequence are unimodal?
3. Let n be a fixed positive integer, and let a_k be the number of compositions of n into k parts. Is $\{a_n\}_{0 \leq k \leq n}$ a unimodal sequence?

9.2 Log-concavity

Interestingly, there is a property of sequences, called *log-concavity*, that is stronger than unimodality, yet often is easier to prove.

9.2.1 Log-concavity implies unimodality

Let n be a fixed positive integer, and let $a_{n,k}$ denote the number of involutions of length n that have k 2-cycles. It is straightforward to compute (see Example 4.36) that

$$a_{n,k} = \frac{n!}{k!(n-2k)! \cdot 2^k}. \tag{9.2}$$

Consider the sequence $a_{n,0}, a_{n,1}, \cdots, a_{n,\lfloor n/2 \rfloor}$ for the first few values of n. See Figure 9.4.

From these data, it seems that the sequences $\{a_{n,k}\}_{0 \leq k \leq \lfloor n/2 \rfloor}$ are unimodal for each fixed n. However, proving this directly seems difficult, even if there is an exact formula for the numbers $a_{n,k}$, since it is not clear where the *peak* of the sequences is. Indeed, for $n = 4$ and $n = 6$, the peak is not at the end of the sequence, while for the other values of n, it is, though sometimes tying the next-to-last element of the sequence. Not knowing where the peak is caused difficulty, since without the position of the peak we cannot prove unimodality by proving that the sequence increases until it reaches the peak, and decreases afterward.

In similar situations, it is often easier to prove a stronger property of sequences.

n				
1	1			
2	1	1		
3	1	3		
4	1	6	3	
5	1	10	15	
6	1	15	45	15
7	1	21	105	105

Figure 9.4
The sequences $\{a_{n,k}\}_{0\leq k\leq \lfloor n/2\rfloor}$, for $n \leq 7$.

Definition 9.6 *The sequence $d_0, \cdots, d_m$ of positive real numbers is called* log-concave *if, for all $k \in [m-1]$,*

$$d_{k-1}d_{k+1} \leq d_k^2. \tag{9.3}$$

The terminology "log-concave" is easy to explain. If a sequence is log-concave, then the sequence of the *logarithms* of its elements (in any fixed base) is *concave*. Indeed, let us take the logarithm of both sides of (9.3) in some fixed base and divide by 2, to get

$$\frac{\log d_{k-1} + \log d_{k+1}}{2} \leq \log d_k,$$

for all $k \in [m-1]$. That is, the graph of the sequence $\log d_k$ is *concave*; in other words, $\log d_k$ is above the line connecting $\log d_{k-1}$ and $\log d_{k+1}$. See Figure 9.5 for an illustration.

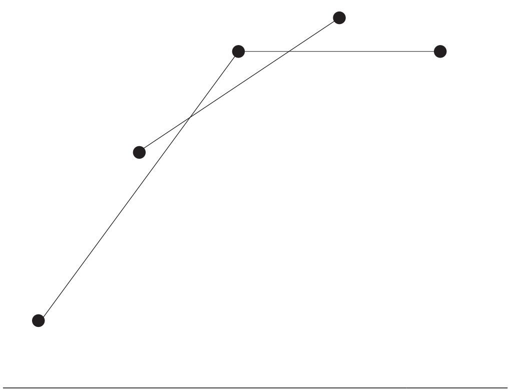

Figure 9.5
A concave sequence. Each element is larger than the average of its two neighbors.

The crucial property of log-concavity is presented in the following proposition.

Proposition 9.7 *If a sequence of positive real numbers is log-concave, then it is unimodal.*

Proof: On one hand, the sequence $d_0, d_1, \cdots, d_m$ is unimodal if and only if it is true that, once the ratio d_{k+1}/d_k dips below 1, it stays there. On the other hand, the sequence $d_0, d_1, \cdots, d_m$ is log-concave if and only if the sequence d_{k+1}/d_k is monotone decreasing. (This can be seen by dividing both sides of (9.3) by $d_k d_{k+1}$.) Because the second condition is stronger, our statement is proved. $\diamondsuit$

It follows from the previous proof that log-concavity is a *strictly* stronger property than unimodality. Indeed, the sequence $1, 1, 2$ is unimodal, but not log-concave.

Let us use our fresh knowledge to prove that the sequence $\{a_{n,k}\}_k$ is unimodal.

Proposition 9.8 *Let $a_{n,k}$ be defined as above. Then, for any fixed n, the sequence $a_{n,0}, a_{n,1}, \cdots, a_{n,\lfloor n/2 \rfloor}$ is log-concave and therefore unimodal.*

Proof: Using Formula 9.2, we get

$$b_{k+1} = \frac{a_{n,k+1}}{a_{n,k}} = \frac{\frac{n!}{(k+1)!(n-2k-2)!2^{k+1}}}{\frac{n!}{k!(n-2k)!\cdot 2^k}} = \frac{(n-2k)(n-2k-1)}{2(k+1)},$$

and

$$b_k = \frac{a_{n,k}}{a_{n,k-1}} = \frac{\frac{n!}{k!(n-2k)!2^k}}{\frac{n!}{(k-1)!(n-2k+2)!\cdot 2^{k-1}}} = \frac{(n-2k+1)(n-2k+2)}{2k}.$$

Therefore, comparing the preceding two equations, we get

$$\frac{b_{k+1}}{b_k} = \frac{n-2k-1}{n-2k+1} \cdot \frac{n-2k}{n-2k+2} \cdot \frac{k}{k+1} < 1,$$

since each of the three factors in the middle is less than 1.

Therefore, the ratio $\frac{a_{n,k+1}}{a_{n,k}}$ is monotone decreasing, which is equivalent to the log-concavity of the sequence $a_{n,k}$. $\diamondsuit$

Once again, we would like to point out that, in this proof of log-concavity, we did *not* need to know where the *peak* of the sequence was.