The reader can easily verify that this representation is bijective. That is, the set of n-plant lines containing k flowers has as many elements as the set of northeastern lattice paths consisting of n steps, k of which are north steps in which north steps are labeled with elements of $[2]$, and east steps are labeled with elements of $[3]$. Therefore, we can now continue our argument in the language of these lattice paths, instead of plant lines.

The rest of the proof is analogous to the above proof of log-concavity of the binomial coefficients. Define L_k just as in that proof, except that the lattice paths are now labeled. Then define the map $f : L_{k-1} \times L_{k+1} \rightarrow L_k \times L_k$ as in that proof as well, that is, by swapping the parts of paths that come after their first intersection point. The fact that the paths are labeled does not change anything, since the validity of a label only depends on the direction of the step to which the label is assigned. So f is an injection by the same argument, and log-concavity is proved.

Quick Check

1. Let $f_{n,m}(x) = (1+x)^n \cdot I_m(x)$. Prove that $f_{n,m}(x)$ is a log-concave polynomial.
2. Recall that $A(n,k)$ denotes an Eulerian number. Prove that the sequence $\{A(n,2)\}_{n\geq 2}$ is log-concave.
3. Let $D(n)$ denote the number of derangements of length n. Prove that the sequence $\{D_n\}_{n\geq 2}$ is log-convex, that is, the inequality $D(n+1)D(n-1) \geq D_n^2$ holds, for all $n \geq 3$.

9.3 The real zeros property

There is a related property of sequences that is even stronger than log-concavity. We say that the sequence $a_0, a_1, \cdots, a_n$ of positive real numbers *has real zeros only* if the polynomial $\sum_{k=0}^n a_k x^k$ has real roots only. Note that, in this case, these roots cannot be positive, since for $x > 0$, we have $\sum_{k=0}^n a_k x^k > 0$.

Example 9.21 *For any fixed positive integer n, the sequence of binomial coefficients $\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$ has real zeros only.*

The reader will be asked to prove this simple statement in Supplementary Exercise 5.

The following theorem is the reason that we discuss the real zeros property in this chapter, but it is not the only interesting consequence of the real zeros property. Other interesting consequences will be mentioned in the Notes section.

Theorem 9.22 *If a sequence of positive real numbers has real zeros only, then it is log-concave.*

Note that the converse is not true. The sequence $1, 1, 1$ is a counterexample.

In order to prove Theorem 9.22, we will need the following classic result from real analysis.

Theorem 9.23 (Rolle's theorem) *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Assume furthermore that $f(a) = f(b) = y$. Then there exists a point $t \in [a, b]$ so that $f'(t) = 0$.*

Proof: As f' is continuous on the closed finite interval $[a, b]$, it follows that f has a maximum and a minimum value on this interval $[a, b]$. If both these values are equal to y, then f is constant on $[a, b]$, and $f'(x) = 0$ for all $x \in (a, b)$.

If f has a minimum or maximum value c on $[a, b]$ that is not equal to y, then let us assume without loss of generality that c is a maximum value, and that $f(t) = c$, where $t \in (a, b)$. Then $f'(t) = 0$ because f has a maximum in t. $\diamondsuit$

The following special case is the one that is most often used in combinatorial applications.

Corollary 9.24 *If a polynomial $r(x)$ with real coefficients has real zeros only, then so does its derivative $r'(x)$.*

Proof: First assume for simplicity that all n roots of r are *distinct*. Then, by Rolle's theorem, applied with $y = 0$, the polynomial r' must have a root between any two consecutive roots of r. This provides $n - 1$ real roots, so all $n - 1$ roots of $r - 1$ are real. See Figure 9.8 for an illustration.

Now let us assume r has some repeated roots as well. Let $a_1, a_2, \cdots, a_k$ be the multiplicities of the k distinct roots of r, with $\sum_{i=1}^{k} a_i = n$. It follows from the derivation rule of products that, if r_i is a root of r with multiplicity a_i, then r_i is a root of r' with multiplicity $a_i - 1$. Therefore, the roots r_i of r will provide a total of $\sum_{i=1}^{k}(a_i - 1) = n - k$ real roots of r'. Then, we can apply Rolle's theorem for each pair of consecutive distinct roots of r, and we get $k - 1$ additional real roots. This completes the proof. $\diamondsuit$

We can now prove Theorem 9.22.

Proof: (of Theorem 9.22) Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients that has real zeros only. Consider the two-variable polynomial

$$q(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i} = p(x/y) y^n.$$

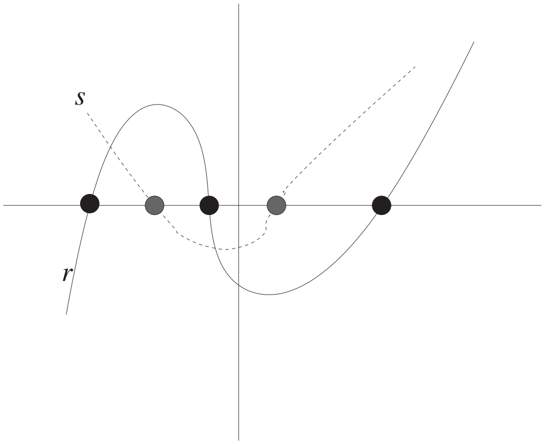

Figure 9.8
The roots of r and $s = r'$ are real and interlacing.

Then q may have roots (x, y) which contain complex numbers, but then, even in these roots, the ratio x/y must be real. Indeed, the above decomposition shows that, if $q(x, y) = 0$ and x or y are not real, then $p(x/y)$ must be 0, forcing the ratio of x and y to be real.

Now, for any fixed real x, we can look at the function $q(x, y)$ as a function of y. This function has real zeros. Therefore, if we differentiate this function with respect to y, the obtained function will have real zeros only, by Corollary 9.24. This will then imply that, for any roots (x, y) of $\partial q/\partial y$, the ratio x/y has to be real. A similar argument can be applied to x instead of y. Iterating this argument shows that the partial derivatives $\partial^{a+b}q/\partial x^a \partial y^b$ also have real zeros only, as long as they are not identically zero, that is, as long as $a + b \leq n - 1$.

Since we want to prove that $a_j^2 \geq a_{j-1}a_{j+1}$, that is, we want to prove an inequality involving three parameters, it is plausible to look for quadratic polynomials deduced from $q(x, y)$. Such polynomials can be obtained by differentiating q a total of $n-2$ times. So let $a = j-1$, let $b = (n-2)-(j-1) = n-j-1$, and let us consider

$$T(x, y) = \partial^{a+b}q/\partial x^a \partial y^b = \partial^{n-2}q/\partial x^{j-1}\partial y^{n-j-1}.$$

Fine, the reader might say at this point, but what can we tell about such complicated partial derivatives? Fortunately, quite a lot. Note that $\partial x^m/\partial x^{j-1} = 0$ unless $m \geq j-1$, and $\partial y^s/\partial y^{n-j-1} = 0$ unless $s \geq n-j-1$. So the only terms of $T(x, y)$ that do not vanish come from the terms of $q(x, y)$ in which $i \in [j-1, j+1]$. This leads to

$$\begin{aligned} T(x, y) &= \frac{(j-1)!}{2} a_{j-1}(n-j+1)!y^2 + a_j j!(n-j)!xy \\ &+ \frac{(j+1)!}{2} a_{j+1}(n-j-1)!x^2. \end{aligned}$$

We have seen that $T(x,y)$, as a partial derivative of $q(x,y)$, has real zeros only, in the sense that, in any root (x,y) of this equation, the ratio x/y has to be real. On the other hand, $T(x,y)/y^2$ is also a quadratic polynomial in x/y; therefore, the fact that it has real zeros only is equivalent to the fact that its discriminant is nonnegative, that is,

$$[a_j j!(n-j)!]^2 - a_{j-1}a_{j+1}(n-j+1)!(n-j-1)!(j-1)!(j+1)! \geq 0,$$

$$a_j^2 \geq a_{j-1}a_{j+1} \cdot \frac{n-j+1}{n-j}\frac{j+1}{j}, \tag{9.4}$$

which is even stronger than the inequality $a_j^2 \geq a_{j-1}a_{j+1}$, which was to be proved. ♢

The following example shows the power of Theorem 9.22.

Example 9.25 *For all positive integers n, the sequence $\{c(n,k)\}_{1\leq k\leq n}$ of signless Stirling numbers of the first kind has real zeroes only and is therefore log-concave.*

Solution: We have seen in Theorem 4.21 that

$$\sum_{k=1}^{n} c(n,k)x^k = (x+n-1)(x+n-2)\cdots(x+1)x.$$

Therefore, the zeroes of the sequence are $0, -1, \cdots, -n+1$. ♢

So, using Theorem 9.22, we could prove the log-concavity of the sequence $\{c(n,k)\}_{1\leq k\leq n}$ quite effortlessly. Without Theorem 9.22, it is considerably more difficult to prove that log-concavity result. Nevertheless, it is always interesting to find a purely combinatorial proof of a combinatorial fact. Such a proof, that is, an injective proof of the log-concavity of the sequence $\{c(n,k)\}_{1\leq k\leq n}$, was given by Bruce Sagan in [64]. In that same paper, Sagan also provided a combinatorial proof of the fact that the sequence $\{S(n,k)\}_{1\leq k\leq n}$ is log-concave. This sequence is also known to have real zeros, but the proof of that fact is not simple.

While many naturally defined combinatorial sequences, such as sequences of binomial coefficients, Stirling numbers, and Eulerian numbers, are known to have real zeros, there are a few intriguing conjectures in this area. The interested reader should consult [20] for these. The reader is also encouraged to read the upcoming Notes for further applications of the real zeros property.

Quick Check

1. Let n be a fixed positive integer, and let $a_k = \binom{n}{k}2^k$. Prove that the sequence $\{a_k\}_{0\leq k\leq n}$ is log-concave.