(B) Let $P(x)$ and $Q(x)$ be two polynomials with positive coefficients. If the sequence of coefficients of $P(x)$ is log-concave, and the sequence of coefficients of $Q(x)$ is log-concave, then the sequence of coefficients of $P(x)Q(x)$ is also log-concave.

9.6 Exercises

1. Prove that, for any fixed positive integer n and any $q > 1$, the sequence $\begin{bmatrix}\mathbf{n}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{n}\\\mathbf{1}\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{n}\\\mathbf{n}\end{bmatrix}$ is log-concave. (The Gaussian coefficients $\begin{bmatrix}\mathbf{n}\\\mathbf{k}\end{bmatrix}$ are defined in Exercise 29 of Chapter 4.)
2. (a) Recall that in (5.6) we defined the Narayana numbers $N(n,k)$ by setting $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. Prove that for any fixed positive integer n, the sequence $N(n,0), N(n,1), \cdots N(n,n-1)$ of Narayana numbers is log-concave.

   (b) Prove that, for any fixed positive integer k, the infinite sequence $N(k+1,k), N(k+2,k), \cdots$ is log-concave.
3. We call a sequence $a_0, a_1, \cdots$ of positive real numbers *log-convex*, if for all $k \geq 1$, we have
$$a_{k-1}a_{k+1} \geq a_k^2.$$
   Prove by a combinatorial argument that the sequence $c_0, c_1, c_2, \cdots$ of Catalan numbers is log-convex.
4. Let us label the vertices of a square grid as shown in Figure 9.9, and let F be a Ferrers shape in the northwestern corner of this grid.

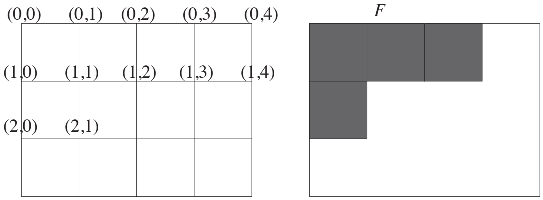

Figure 9.9
How to label the nodes of the grid and how to place the Ferrers shape F.

Let $M_F(m,n) = M(m,n)$ be the number of northeastern lattice paths from $(m,0)$ to $(0,n)$ that *do not go inside* F.

(a) Prove that
$$M(m,n+1)M(m+1,n) \leq M(m,n)M(m+1,n+1). \quad (9.5)$$

(b) + Prove that

$$M(m-1,n+1)M(m+1,n+1) \leq M(m,n+1)^2. \qquad (9.6)$$

(c) Conclude that

$$M(m-1,n+1)M(m+1,n) \leq M(m,n)M(m,n+1). \qquad (9.7)$$

5. Keep the notation of the previous exercise.
   (a) Prove that $M(m+1,n-1)M(m,n+1) \leq M(m,n)M(m+1,n)$.
   (b) Conclude that $M(m+1,n-1)M(m-1,n+1) \leq M(m,n)^2$, that is, that the sequence $\{M(i,n+m-i)\}_{0\leq i\leq m+n}$ is log-concave.
   (c) Explain how this result generalizes the fact that the binomial coefficients $\binom{r}{k}$ form a log-concave sequence for any fixed r.
   (d) Must the sequence $\{M(i,n+m-i)\}_{0\leq i\leq m+n}$ have real zeroes only?
6. + Find a more direct combinatorial proof of part (b) of the previous exercise, that is, of the log-concavity of the sequence

$$\{M(i,n+m-i)\}_{0\leq i\leq m+n}.$$

7. We call the sequence $a_0, a_1, \cdots, a_n$ of positive real numbers *strongly log-concave*, if for $1 \leq i \leq n-1$, we have

$$(n-(i-1))a_{i-1} \cdot (i+1)a_{i+1} \leq (n-i)a_i \cdot ia_i.$$

   Prove that, if the sequence $a_0, a_1, \cdots, a_n$ of positive real numbers has real zeros only, then it is strongly log-concave.
8. Is it true that if a finite sequence of positive real numbers is strongly log-concave, then it has real zeros only?
9. Let G be a bipartite graph with color classes A and B. For any subset $X \subseteq A$, let $N(X)$ denote the set of vertices in B that have a neighbor in X.

   Let us say that *A has a perfect matching into B* if there are $|A|$ vertex-disjoint edges in G, or in other words, if each vertex of A can be matched to an adjacent vertex of B.

   A classic theorem of graph theory, *Philip Hall*'s theorem, says that A has a perfect matching into B if and only if $|X| \leq |N(X)|$ for all $X \subseteq A$. Use this theorem to prove that the sequence $\{\binom{n}{k}\}_{0\leq k\leq n}$ is unimodal.
10. + Let n be a fixed positive integer. Find a combinatorial (that is, injective) proof for the unimodality of the sequence of Narayana numbers $N(n,0), N(n,1), \cdots, N(n,n-1)$. *Hint: Try to use the reflection principle.*