11. Find a combinatorial proof of the log-concavity of the infinite sequence $S(2,2), S(3,2), S(4,2), \cdots$. Recall that $S(n,k)$ denotes a Stirling number of the second kind, that is, it is the number of partitions of $[n]$ into k blocks.

9.7 Solutions to exercises

1. Let us compute the ratio $\frac{\left[{\mathbf{n} \atop \mathbf{k}}\right]}{\left[{\mathbf{n} \atop \mathbf{k-1}}\right]}$. We get the fraction

$$\frac{q^{n-k}-1}{(q^k-1)q^{k-1}},$$

which decreases as k increases. Indeed, the numerator decreases and the denominator increases.

2. (a) Again, look at the ratio

$$\begin{aligned}\frac{N(n,k)}{N(n,k-1)} &= \frac{\binom{n}{k+1}}{\binom{n}{k-1}} = \frac{(k-1)!(n-k+1)!}{(k+1)!(n-k-1)!} \\ &= \frac{(n-k+1)(n-k)}{k(k+1)},\end{aligned}$$

which again decreases as k increases.

(b) We need to prove that $N(n+1,k)^2 \geq N(n,k)N(n+2,k)$, that is,

$$\frac{\binom{n+1}{k}^2\binom{n+1}{k+1}^2}{(n+1)^2} \geq \frac{\binom{n}{k}\binom{n}{k+1}\binom{n+2}{k}\binom{n+2}{k+1}}{n(n+2)}.$$

Multiplying both sides by $(k!(k+1)!)^2$, we get

$$((n)_k(n+1)_k)^2 \geq (n-1)_k(n)_k(n+1)_k(n+2)_k,$$

$$n(n+2-k) \geq (n+2)(n-k).$$

The last inequality holds, either by routine simplification to $0 \geq -2k$, or by noting that the sequence $1, 2, 3, \cdots$ is log-concave.

3. Let $\mathcal{C}_n$ be the set of northeastern lattice paths from $(0,0)$ to (n,n) that do not go above the main diagonal. Then $\mathcal{C}_n = c_n$. An injection $f_k : \mathcal{C}_n \times \mathcal{C}_n \to \mathcal{C}_{n-1} \times \mathcal{C}_{n+1}$ can be defined as follows: If $(p,q) \in \mathcal{C}_n \times \mathcal{C}_n$, translate q by $(-1,-1)$ so it becomes a path q' from $(-1,-1)$ to $(n-1,n-1)$. From here on, the proof is similar to the lattice path proof of the log-concavity of the binomial coefficients that we saw in the text. That is, let X be the first intersection of p and q',

and swap the parts of p and q' that come after X. See Figure 9.10 for an example. This results in a path from $(0,0)$ to $(n-1,n-1)$ and a path from $(-1,-1)$ to (n,n). Since this map is an injection, our claim is proved.

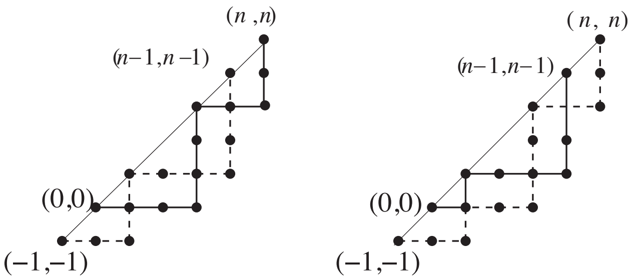

Figure 9.10
Our injection $f_k : \mathcal{C}_n \times \mathcal{C}_n \to \mathcal{C}_{n-1} \times \mathcal{C}_{n+1}$.

4. The results of this exercise and the following one were all found during research efforts by the author and Bruce Sagan, published in [17], while finding a combinatorial proof of Rodica Simion's conjecture claiming that the sequence $\{M(i, n+m-i)\}_{0\leq i\leq m+n}$ is unimodal. That conjecture, in the stronger form that the sequence is in fact log-concave, now has four proofs, though the other two are not combinatorial.

   (a) This can be done by a simple lattice path argument. Let p be a path counted by $N(m, n+1)$, and let q be a path counted by $N(m+1, n)$. Then these two paths must intersect, and swapping them at their first intersection point, as seen in Example 9.20, we get a pair of paths counted by $M(m,n)\cdot M(m+1, n+1)$. This map is an injection. See Figure 9.11 for an illustration.

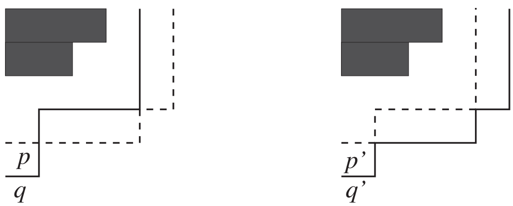

Figure 9.11
Swapping lattice paths at their first intersection points.

(b) This part is trickier, since a path counted by $M(m-1,n)$ and a path counted by $M(m+1,n)$ do not necessarily intersect before their endpoint.

Let p be a path counted by $M(m-1,n)$, and let q be a path counted by $M(m+1,n)$. Find the first point P on p that is exactly one unit below a point Q of q. Such a point has to exist since the vertical distance of p and q changes from 2 to 0. Then move the part of p preceding P one unit to the south and attach it to the second part of Q, while moving the part of q preceding Q one unit to the north and attaching it to the second part of p. See Figure 9.12 for an illustration. This constitutes

Figure 9.12
Swapping lattice paths when their vertical distance is 1.

"swapping" the initial segments of p and q and provides the needed injection. The reader is invited to verify the details, that is, that this map is indeed an injection and that this is a valid map (that is, that the obtained paths will never go inside F).

(c) Multiply the inequalities proved in parts (a) and (b) together and simplify.

5. (a) Let F^c be the *conjugate* of the Ferrers shape F. Because the result of part (c) of the previous exercise holds for all Ferrers shapes, it holds for F^c as well. On the other hand, we have $M(a,b) = M_{F^c}(b,a)$. Therefore, if we take (9.7) for F^c instead of F, and then apply the mentioned symmetry, we get

$$M(n+1,m-1)M(n,m+1) \leq M(n,m)M(n+1,m).$$

Now switch m and n.

(b) Take the product of (9.7) and the inequality proved in part (a) and simplify.

(c) If F is the empty Ferrers shape, then the sequence that we prove to be log-concave is the sequence $\{\binom{n+m}{i}\}_{0\leq i\leq m+n}$.

(d) No, it will not, even in very simple cases. Let F consist of one box, and let $m+n=4$. This will lead to the polynomial $3x^3+5x^2+3x$, which has two complex roots.

6. Take a path p that is counted by $M(m-1, n+1)$ and a path q that is counted by $M(m+1, n-1)$. We will map this pair injectively onto a pair (p', q') in which both paths belong to the set enumerated by $M(m, n)$.

   In order to do this, we will cut our paths p and q into *three* parts, instead of two, and we will swap their *middle* parts.

   Similarly to what we have seen in the solution of part (b) of Exercise 4, let P and Q be the first points on p and q so that P is one unit above q. Furthermore, let $P*$ and $Q*$ be the last points on p and q so that $P*$ is one unit to the east of $Q*$. Then the defined four points split p into segments p_1, p_2, p_3 and q into q_1, q_2, q_3. Then we map (p, q) into the pair $(p_1q_2p_3, q_1p_2q_3)$, where p_1 is moved one unit south, p_3 is moved one unit west, q_1 is moved one unit north, and q_3 is moved one unit east. See Figure 9.13 for an example. The reader

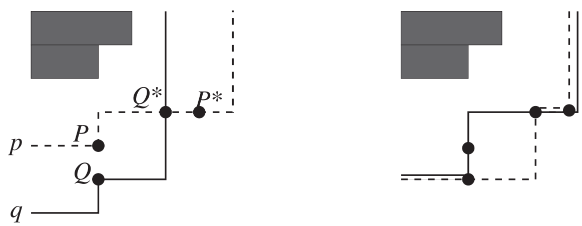

Figure 9.13
We swap the middle parts of p and q.

   is asked to verify that this map is injective and that the obtained paths do not go inside F.

7. This follows from (9.4) by routine rearrangements.

8. No, that is false. A counterexample is the sequence 1, 3, 3.

9. Let $k \leq \lfloor (n-1/2) \rfloor$. Define the graph G to be the bipartite graph whose vertices correspond to the k-element and $(k+1)$-element subsets of $[n]$. Let vertices x and y be adjacent if $x \subset y$, where *strict* containment is required. Then G is bipartite, and its color classes A and B have $\binom{n}{k}$ and $\binom{n}{k+1}$ vertices, respectively, since they correspond to the collections of k-element and $(k+1)$-element subsets of $[n]$. Our claim that $\binom{n}{k} \leq \binom{n}{k+1}$ will be proved if we can show that A has a perfect matching into B.

   It suffices to show that the conditions of Philip Hall's theorem hold. To that end, note that G is a *regular* graph. Indeed, each vertex of A has degree $n-k$, and each vertex of B has degree $k+1$. Because we know that $k \leq \lfloor (n-1/2) \rfloor$, we see that $n-k \geq k+1$, so the vertices in A have a degree at least as high as the vertices in B.

Now look at the induced subgraph H of G whose vertices are the vertices of S and $N(S)$, for some $S \subseteq A$. As vertices of S do not lose any neighbors when we restrict our attention to this subgraph, their degrees are still at least as high as any degree in $N(S)$. Since the sum of all degrees of vertices in S and $N(S)$ must agree, the inequality $|S| \leq |N(S)|$ follows. Therefore, A has a perfect matching into B, and so $|A| \leq |B|$. The rest of unimodality follows from the symmetry of our sequence.

10. We have seen in Exercise 23 that the Narayana number $N(n,k)$ counts unlabeled binary trees on n vertices with k right edges. Let $\mathcal{N}(n,k)$ be the set of such trees, and let $k \leq (n-1)/2$. We will construct an injection $f : \mathcal{N}(n,k) \rightarrow \mathcal{N}(n,k+1)$.

    Let $T \in \mathcal{N}(n,k)$. Let us define a total ordering on the set of vertices of T as follows: Order the vertices first according to their distance from the root (furthest vertices first), and then left to right. Let T_i be the induced subgraph of the first i vertices of T; then T_i is a forest. Furthermore, T_i has either the same number of right edges as T_{i-1} or one more. Since T_0 has the same number of left and right edges, and T_n has at least one more right edge than left edge, the previous sentence implies that there has to be a (smallest) index j so that T_j contains *exactly* one more right edge than left edge.

    Now reflect each component of T_j through a vertical axis that contains its root. Let the obtained tree be $f(T)$. Then $f(T)$ has one more left edge than T, and $f : \mathcal{N}(n,k) \rightarrow \mathcal{N}(n,k+1)$. Indeed, to find the preimage of a tree $T' \in \mathcal{N}(n,k+1)$, just find the smallest index j so that T'_j has one more left edge than right edge (if such an index exists), and reflect each component of T'_j through a vertical axis that contains its root. Figure 9.14 shows an example.

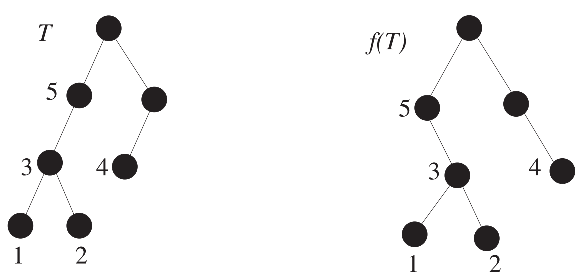

Figure 9.14
In this example, $j = 5$, since T_j has two left edges and one right edge.

11. A partition of $[n]$ into an *ordered pair* of two blocks can be described by a binary vector of length n whose ith coordinate is 1 if i is part