1. The initial step: If $n = 1$, then the statement says that $0 = 0$, so the statement is true.
2. The induction step: Let us assume that the statement is true for $n = k$, that is,

$$\sum_{i=1}^{k} i(i-1) = \frac{(k+1)k(k-1)}{3} \tag{A.2}$$

holds. We need to prove that then the statement also holds for $n = k+1$, that is,

$$\sum_{i=1}^{k+1} i(i-1) = \frac{(k+2)(k+1)k}{3}. \tag{A.3}$$

In other words, we need to show that (A.2) implies (A.3). In order to achieve that goal, it clearly suffices to show that the *difference* of (A.3) and (A.2) holds, since then we can add that difference to both sides of (A.2) and get (A.3). On the other hand, this difference is the equation

$$(k+1)k = (k+1)k \cdot \frac{(k+2)-(k-1)}{3},$$

which is clearly an identity. This completes the induction step.

Therefore, by induction, (A.1) holds for all positive integers n. $\diamondsuit$

A.2 Strong induction

Sometimes knowing that a statement is true today is not sufficient to conclude that the statement will be true tomorrow; sometimes we need to know that the statement has been true *every day up to now* in order to allow such a conclusion. Similarly, sometimes we need to know that a statement is true for all values $k \leq n$ in order to conclude that it is true for $n+1$ as well. If that is the case, and the statement is true for $n = 1$, then it must be true for all values of n. Indeed, from the fact that it is true for $n = 1$, it follows that it is true for $n = 2$, since 1 is the only positive integer smaller than 2. Then, as the statement is true for $n = 1$ and $n = 2$, it is true for $n = 3$, and then, as it is true for $n = 1$ and $n = 2$ and $n = 3$, it is true for $n = 4$, and so on.

This way of proving statements for all n using a *stronger* induction hypothesis is called the method of *strong induction*.

Example A.2 *Let $a_0 = 1$, and let*

$$a_n = \sum_{i=0}^{n-1} 2a_i \tag{A.4}$$

if $n \geq 1$. Prove that then $a_n = 2 \cdot 3^{n-1}$ for all positive integers n.

Proof:

1. The initial step: For $n = 1$, formula (A.4) yields $a_1 = 2 \cdot a_0 = 2$, agreeing with the formula to be proved.
2. The induction step: Let us assume that the statement holds for all $k \leq n$. Then (A.4) yields

$$\begin{aligned} a_{n+1} &= \sum_{k=0}^{n} 2a_k \\ &= 2(1 + 2 \cdot \sum_{k=1}^{n} 3^{k-1}) \\ &= 2 + 4 \cdot \sum_{k=1}^{n} 3^{k-1} \\ &= 2 + 4 \cdot \frac{3^n - 1}{2} \\ &= 2 \cdot 3^n. \end{aligned}$$

   So our statement holds for $k = n + 1$ as well.

Therefore, by strong induction, the statement holds for all positive integers n. $\diamondsuit$

Induction has many variations. The smallest allowed value of n can be different from 1, in which case the initial step needs to be changed accordingly, or there could be many variables. The reader is encouraged to consult the references mentioned at the beginning of this appendix and practice inductive proofs if the reader has not done so before.

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